Differential Geometry

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Abstract

The theory of real vector spaces, norms and derivatives of functions between normed vector spaces as required for formal modelling of some physical theories.

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Contents

To Do

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References

[1] S.W. Hawking and G.F.R. Ellis. *The large scale structure of space-time.* Cambridge University Press, 1973.

[2] LEMMA1/HOL/WRK066. *Mathematical Case Studies: Basic Analysis.* R.D. Arthan, Lemma 1 Ltd., `rda@lemma-one.com`.

[3] LEMMA1/HOL/WRK068. *Mathematical Case Studies: Some Group Theory.* R.D. Arthan, Lemma 1 Ltd., `rda@lemma-one.com`.

1 INTRODUCTION

This document contains some first approaches to formalising differential fer use in the formalisation of Physics. It is now of little interest, since my inclination now is to work with geometric algebra and geometric calculus. However, though Rob Arthan has now formalised "The Geometric Algebra", there has been no progress (so far as I am aware) on differential geometry based on geometric algebra.

Create new theory "diffgeom", parents being "group_egs" from [3] and "analysis" from [2].

SML

```
open_theory "rbjmisc";
open PreConsisProof; open UnifyForwardChain; open Trawling;
force_new_theory "diffgeom";
new_parent "geomalg";
set_merge_pcs["basic_hol1", "'sets_alg", "'ℝ", "'savedthm_cs_∃_proof"];
set_flag ("pp_use_alias", false);
```

2 MISCELLANEA

2.1 The Schwartz Inequality and Triangle Theorems

3 VECTOR SPACES

3.1 Signature for Real Vector Spaces

A candidate for being a vector space is something with the signature of a group together with an ℝ-action on the elements.

HOL Labelled Product

```
RVS
    Group_RVS  :'a GROUP;
    Scale_RVS  :ℝ → 'a → 'a
```

SML

```
declare_alias("Grp", ⌜Group_RVS⌝);
declare_alias("Scale", ⌜Scale_RVS⌝);
```

HOL Constant

$$\begin{array}{l}
\boldsymbol{Plus_V} : {'a} \to {'a} \to {'a}\ RVS \to {'a};\\
\boldsymbol{Minus_V} : {'a} \to {'a}\ RVS \to {'a};\\
\boldsymbol{Subtract_V} : {'a} \to {'a} \to {'a}\ RVS \to {'a};\\
\boldsymbol{0_V} : {'a}\ RVS \to {'a};\\
\boldsymbol{Scale_V} : \mathbb{R} \to {'a} \to {'a}\ RVS \to {'a}
\end{array}$$

$$\begin{array}{l}
\forall\ R \bullet\quad (\forall v\ w \bullet\ Plus_V\ v\ w\ R = (v.w)(Grp\ R))\\
\quad \land\ (\forall v \bullet\ Minus_V\ v\ R = (v\ \tilde{})(Grp\ R))\\
\quad \land\ (\forall v\ w \bullet\ Subtract_V\ v\ w\ R = (v.(w\ \tilde{})(Grp\ R))(Grp\ R))\\
\quad \land\ 0_V\ R = Unit\ (Grp\ R)\\
\quad \land\ (\forall x\ v \bullet\ Scale_V\ x\ v\ R = (Scale\ R)\ x\ v)
\end{array}$$

SML

```
declare_infix(310, "*_s");
declare_alias("+", ⌜Plus_V⌝);
declare_alias("~", ⌜Minus_V⌝);
declare_alias("−", ⌜Subtract_V⌝);
declare_alias("*_s", ⌜Scale_V⌝);
```

3.2 Vector Space Laws

For simplicity in using the theory, we ignore the carrier set of the group component and require the carrier set of a real vector space to be the universe of the type of the elements. This makes working with subspaces harder theoretically, but that shouldn't matter much for RBJ's application.

HOL Constant

$$\boldsymbol{VS_R} : {'a}\ RVS\ SET$$

$$\begin{array}{l}
\forall\ V \bullet\\
\quad V \in VS_R\\
\Leftrightarrow Grp\ V \in AbelianGroup\\
\land\ Car\ (Grp\ V) = Universe\\
\land\ (\forall x\ v\ w \bullet\ ((x *_s v) V + (x *_s w) V) V = (x *_s (v + w)\ V)\ V)\\
\land\ (\forall x\ y\ v \bullet\ ((x *_s v) V + (y *_s v) V) V = ((x + y) *_s v)\ V)\\
\land\ (\forall x\ y{:}\mathbb{R};\ v \bullet\ (x *_s (y *_s v) V) V = ((x * y) *_s v) V)\\
\land\ (\forall v \bullet\ (\mathbb{NR}\ 1 *_s v) V = v)
\end{array}$$

3.3 Examples

HOL Constant

$\mathbb{R}_{\boldsymbol{RVS}} : \mathbb{R}\ RVS$

$\mathbb{R}_{RVS} = MkRVS\ \mathbb{R}_+\ (\lambda x\ y \bullet\ x * y)$

3.4 Product Spaces

HOL Constant

$\boldsymbol{VectorSpaceProduct} : 'a\ RVS \to 'b\ RVS \to ('a \times 'b)\ RVS$

$\forall\ V{:}'a\ RVS;\ W{:}'b\ RVS \bullet\ VectorSpaceProduct\ V\ W =$
$\quad let\ group = (Grp\ V) * (Grp\ W)$
$\quad and\ action\ (r{:}\mathbb{R})\ (ga,\ gb) = ((r *_s ga)\ V,\ (r *_s gb)\ W)$
$\quad in\ MkRVS\ group\ action$

SML

$declare\_alias$ ("∗", ⌜$VectorSpaceProduct$⌝);

3.5 Linear Mappings

A homomorphism between vector spaces is called a linear mapping and is defined as follows:

HOL Constant

$\boldsymbol{Lin} : 'a\ RVS \times 'b\ RVS \to ('a \to 'b)\ SET$

$\forall\ V\ W\ f \bullet$
$\quad f \in Lin(V,\ W)$
$\Leftrightarrow f \in Homomorphism(Grp\ V,\ Grp\ W)$
$\land\ (\forall x\ v \bullet\ f((x *_s v)\ V) = (x *_s f\ v)\, W)$

HOL Constant

$\boldsymbol{Fun_G} : ('a \to \mathbb{R})\ GROUP$

$Fun_G = MkGROUP\ Universe\ (\lambda f\ g\ a \bullet f\ a + g\ a)\ (\lambda a \bullet \mathbb{N}\mathbb{R}\ 0)\ (\lambda f\ a \bullet \sim(f\ a))$

HOL Constant

$\boldsymbol{Fun_{RVS}} : ('a \to \mathbb{R})\ RVS$

$Fun_{RVS} = MkRVS\ Fun_G\ (\lambda x{:}\mathbb{R};\ f \bullet\ \lambda a{:}'a \bullet x * f\ a)$

HOL Constant

$\mathbb{R}^{\boldsymbol{2}}{}_{\boldsymbol{RVS}} : (\mathbb{R} \times \mathbb{R})\ RVS$

$\mathbb{R}^2{}_{RVS} = \mathbb{R}_{RVS} * \mathbb{R}_{RVS}$

HOL Constant

$\mathbb{R}^{\mathbf{3}}_{\boldsymbol{RVS}} : (\mathbb{R} \times \mathbb{R} \times \mathbb{R})\ RVS$

$\mathbb{R}^{3}{}_{RVS} = \mathbb{R}_{RVS} * \mathbb{R}_{RVS} * \mathbb{R}_{RVS}$

Triples of reals are used for spatial coordinates and various vectors, a type abbreviation is used to make the specifications a little more readable.

SML

```
declare_type_abbrev ("ℝ³", [], ⌜:ℝ×ℝ×ℝ⌝);
```

The following definition names the zero 3-tuple.

HOL Constant

$\mathbf{0}_{\boldsymbol{R3}} : \mathbb{R}^{3}$

$0_{R3} = (0_R,\ 0_R,\ 0_R)$

The following function yeilds the square of the distance between two points in a three dimensional Euclidean space.

SML

```
declare_infix (300, "−tr");
```

HOL Constant

$\$-_{\boldsymbol{tr}} : \mathbb{R}^{3} \to \mathbb{R}^{3} \to \mathbb{R}^{3}$

$\forall x_1\ y_1\ z_1\ x_2\ y_2\ z_2 \bullet$
$\quad (x_1,\ y_1,\ z_1) -_{tr} (x_2,\ y_2,\ z_2) = (x_1 - x_2,\ y_1 - y_2,\ z_1 - z_2)$

SML

```
declare_alias ("−", ⌜$−tr⌝);
declare_infix (300, "+tr");
```

HOL Constant

$\$+_{\boldsymbol{tr}} : \mathbb{R}^{3} \to \mathbb{R}^{3} \to \mathbb{R}^{3}$

$\forall x_1\ y_1\ z_1\ x_2\ y_2\ z_2 \bullet$
$\quad (x_1,\ y_1,\ z_1) +_{tr} (x_2,\ y_2,\ z_2) = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$

SML

```
declare_alias ("+", ⌜$−tr⌝);
declare_infix (310, "*trs");
```

HOL Constant

$\$*_{\boldsymbol{trs}} : \mathbb{R}^{3} \to \mathbb{R} \to \mathbb{R}^{3}$

$\forall x_1\ y_1\ z_1\ r \bullet$
$\quad (x_1,\ y_1,\ z_1) *_{trs} r = (x_1 * r,\ y_1 * r,\ z_1 * r)$

SML

```
declare_alias ("*", ⌜$*_trs⌝);
declare_infix (310, "/_trs");
```

HOL Constant

$\$/_{\boldsymbol{trs}} : \mathbb{R}^3 \to \mathbb{R} \to \mathbb{R}^3$

$\forall x_1\ y_1\ z_1\ r \bullet$
$\quad (x_1,\ y_1,\ z_1)\ /_{trs}\ r = (x_1\ /\ r,\ y_1\ /\ r,\ z_1\ /\ r)$

SML

```
declare_alias ("/", ⌜$/_trs⌝);
```

4 NORMED VECTOR SPACES

4.1 Norms

SML

```
declare_type_abbrev("NORM", ["'a"], ⓣ'a → ℝ⌝);
```

HOL Constant

$\boldsymbol{Norm} : 'a\ RVS \to 'a\ NORM\ SET$

$\forall\ V\ norm \bullet$
$\quad norm \in Norm\ V$
$\Leftrightarrow (\forall v \bullet \mathbb{NR}\ 0 \leqslant norm\ v)$
$\wedge\ (\forall v \bullet norm\ v = \mathbb{NR}\ 0 \Leftrightarrow v = 0_V\ V)$
$\wedge\ (\forall x\ v \bullet norm\ ((x *_s v) V) = Abs\ x * norm\ v)$
$\wedge\ (\forall v\ w \bullet norm\ ((v + w) V) \leqslant norm\ v + norm\ w)$

HOL Constant

$\boldsymbol{NormProduct} : 'a\ NORM \to 'b\ NORM \to ('a \times 'b)\ NORM$

$\forall\ n{:}'a\ NORM;\ m{:}'b\ NORM;\ a{:}'a;\ b{:}'b \bullet$
$\quad NormProduct\ n\ m\ (a,\ b) = Abs(SqrtA((n\ a)\ \hat{}\ 2 + (m\ b)\ \hat{}\ 2))$

SML

```
declare_alias("*", ⌜NormProduct⌝);
```

HOL Constant

$\boldsymbol{Di_R} : \mathbb{R}\ NORM;$
$\boldsymbol{Di_{R2}} : (\mathbb{R} \times \mathbb{R})\ NORM;$
$\boldsymbol{Di_{R3}} : (\mathbb{R} \times \mathbb{R} \times \mathbb{R})\ NORM$

$(\forall r{:}\mathbb{R} \bullet\ Di_R\ r = Abs\ r)$
$\wedge\ Di_{R2} = NormProduct\ Di_R\ Di_R$
$\wedge\ Di_{R3} = NormProduct\ Di_R\ Di_{R2}$

4.2 Normed Vector Spaces

HOL Labelled Product

NVS

$Rvs_{NVS} \quad :'a\ RVS;$

$Norm_{NVS} \quad :'a \to \mathbb{R}$

HOL Constant

$\boldsymbol{Nvs} : 'a\ NVS\ SET$

$$\forall\ N \bullet\ N \in Nvs$$
$$\Leftrightarrow Rvs_{NVS}\ N \in VS_R \land (Norm_{NVS}\ N) \in Norm\ (Rvs_{NVS}\ N)$$

HOL Constant

$\boldsymbol{Plus_N} :\ 'a \to 'a \to 'a\ NVS \to 'a;$
$\boldsymbol{Minus_N} :\ 'a \to 'a\ NVS \to 'a;$
$\boldsymbol{Subtract_N} :\ 'a \to 'a \to 'a\ NVS \to 'a;$
$\boldsymbol{0_N} :\ 'a\ NVS \to 'a;$
$\boldsymbol{Scale_N} :\ \mathbb{R} \to 'a \to 'a\ NVS \to 'a;$
$\boldsymbol{Norm_N} :\ 'a \to 'a\ NVS \to \mathbb{R}$

$$\forall\ N \bullet$$
$$(\forall v\ w \bullet\ Plus_N\ v\ w\ N = Plus_V\ v\ w\ (Rvs_{NVS}\ N))$$
$$\land\ (\forall v \bullet\ Minus_N\ v\ N = Minus_V\ v\ (Rvs_{NVS}\ N))$$
$$\land\ (\forall v\ w \bullet\ Subtract_N\ v\ w\ N = Plus_V\ v\ (Minus_V\ w\ (Rvs_{NVS}\ N))\ (Rvs_{NVS}\ N))$$
$$\land\ 0_N\ N = 0_V\ (Rvs_{NVS}\ N)$$
$$\land\ (\forall x\ v \bullet Scale_N\ x\ v\ N = Scale_V\ x\ v\ (Rvs_{NVS}\ N))$$
$$\land\ (\forall v \bullet Norm_N\ v\ N = Norm_{NVS}\ N\ v)$$

SML

```
declare_alias("+", ⌜Plus_N⌝);
declare_alias("~", ⌜Minus_N⌝);
declare_alias("−", ⌜Subtract_N⌝);
declare_alias("*", ⌜Scale_N⌝);
```

HOL Constant

$\boldsymbol{NvsProduct} : 'a\ NVS \to 'b\ NVS \to ('a \times 'b)\ NVS$

$$\forall\ n:'a\ NVS;\ m:'b\ NVS \bullet$$
$$NvsProduct\ n\ m = MkNVS\ ((Rvs_{NVS}\ n) * (Rvs_{NVS}\ m))\ ((Norm_{NVS}\ n) * (Norm_{NVS}\ m))$$

SML

```
declare_alias("*", ⌜NvsProduct⌝);
```

HOL Constant

$\mathbb{R}_{\boldsymbol{NVS}} : \mathbb{R}\ NVS;$
$\mathbb{R}^{\mathbf{2}}{}_{\boldsymbol{NVS}} : (\mathbb{R} \times \mathbb{R})\ NVS;$
$\mathbb{R}^{\mathbf{3}}{}_{\boldsymbol{NVS}} : (\mathbb{R} \times \mathbb{R} \times \mathbb{R})\ NVS;$
$\mathbb{R}^{\mathbf{4}}{}_{\boldsymbol{NVS}} : (\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})\ NVS$

$$\begin{array}{l} \quad \mathbb{R}_{NVS} = MkNVS\ \mathbb{R}_{RVS}\ Di_R \\ \wedge\ \mathbb{R}^2{}_{NVS} = \mathbb{R}_{NVS} * \mathbb{R}_{NVS} \\ \wedge\ \mathbb{R}^3{}_{NVS} = \mathbb{R}_{NVS} * \mathbb{R}^2{}_{NVS} \\ \wedge\ \mathbb{R}^4{}_{NVS} = \mathbb{R}_{NVS} * \mathbb{R}^3{}_{NVS} \end{array}$$

4.3 Frechet Derivative

In the following read, as so often, $\in$ as "is a": the set in question will have at most one element. In ordinary notation, the last inequality here amounts to $|((f(v + h) - f(v) - D(h))/|h|) < e$.

HOL Constant

$\boldsymbol{FrechetDeriv} : ('a\ NVS) \times ('b\ NVS) \to ('a \to 'b) \to 'a \to ('a \to 'b)\ SET$

$$\begin{array}{l} \forall\ (M{:}'a\ NVS)\ (N{:}'b\ NVS)\ f\ (v{:}'a)\ (D{:}\ 'a \to 'b) \bullet \\ \quad D \in FrechetDeriv(M,\ N)\ f\ v \\ \Leftrightarrow D \in Lin(Rvs_{NVS}\ M,\ Rvs_{NVS}\ N) \\ \wedge\ (\forall e{:}\mathbb{R} \bullet\ \mathbb{NR}\ 0 < e \Rightarrow \exists d \bullet \\ \qquad \mathbb{NR}\ 0 < d \\ \quad \wedge\ \ (\forall h{:}'a \bullet\ \mathbb{NR}\ 0 < Norm_N\ h\ M\ \wedge\ Norm_N\ h\ M < d \Rightarrow \\ \qquad Norm_N\ ((((Norm_N\ h\ M)^{-1}) * (((f((v + h)M) - (f\ v))N) - (D\ h))N)\ N)\ N < e)) \end{array}$$

HOL Constant

$\boldsymbol{FDifferentiable} : ('a\ NVS) \times ('b\ NVS) \to ('a \to 'b) \to 'a \to BOOL$

$$\begin{array}{l} \forall\ (M{:}'a\ NVS)\ (N{:}'b\ NVS)\ f\ (v{:}'a) \bullet \\ \quad FDifferentiable\ (M,\ N)\ f\ v \\ \Leftrightarrow \neg\ FrechetDeriv(M,\ N)\ f\ v = \{\} \end{array}$$

The special case of the derivative in which the domain of the function is $\mathbb{R}_{NVS}$ (representing time perhaps) is defined here. A different convention is adopted for representation of a possibly not everywhere defined derivative.

HOL Constant

$\boldsymbol{VDeriv} : 'a\ NVS \to (\mathbb{R} \to 'a) \to (\mathbb{R} \to 'a)$

$$\begin{array}{l} \forall\ (N{:}'a\ NVS)\ f\ r \bullet \\ \quad let\ D = FrechetDeriv\ (\mathbb{R}_{NVS},\ N)\ f\ r \\ \quad in\ \neg D = \{\} \Rightarrow (\lambda r' \bullet\ (r' * (VDeriv\ N\ f\ r))\ N) \in D \end{array}$$

HOL Constant

$\boldsymbol{VNthDeriv} : \mathbb{N} \to ('b\ NVS) \to (\mathbb{R} \to 'b) \to (\mathbb{R} \to 'b)$

$\forall\ (n{:}\mathbb{N});\ N{:}'b\ NVS;\ f\ {:}\mathbb{R} \to 'b \bullet$
$\quad VNthDeriv\ 0\ N\ f = f$
$\land \quad VNthDeriv\ (n+1)\ N\ f =$
$\qquad let\ f' = VDeriv\ N\ f$
$\qquad in\ VNthDeriv\ n\ N\ f'$

HOL Constant

$\boldsymbol{EDiff} : ('a\ NVS) \times ('b\ NVS) \to ('a \to 'b) \to ('a \to 'a \to 'b) \to BOOL$

$\forall\ N\ M\ f\ df \bullet$
$\quad EDiff\ (N,\ M)\ f\ df$
$\Leftrightarrow \forall v \bullet\ df\ v \in FrechetDeriv\ (N,\ M)\ f\ v$

4.4 The Natural Topology Over a Normed Vector Space

HOL Constant

$NVSTopology{:}\ 'a\ NVS \to 'a\ SET\ SET$

$\forall v{:}'a\ NVS \bullet\ NVSTopology\ v = \{vs{:}\ 'a\ SET \mid \forall x{:}'a \bullet\ x \in vs \Rightarrow$
$\quad \exists \xi \bullet\ \forall y{:}'a \bullet\ Norm_N\ ((Subtract_N\ y\ x)\ v)\ v <_R \xi \Rightarrow y \in vs\}$

5 INNER PRODUCT SPACES

5.1 Inner Products

SML

```
declare_type_abbrev("IP", ["'a"], ⌜:'a → 'a → ℝ⌝);
```

SML

```
declare_infix(310, "*_V");
declare_infix(300, "+_V");
declare_infix(310, "._i");
```

HOL Constant

$\boldsymbol{InnerProduct} : {}'a\ RVS \to {}'a\ IP\ SET$

$\forall\ V{:}'a\ RVS;\ \$.{}_i{:}\ 'a \to 'a \to \mathbb{R} \bullet$
$\quad \$.{}_i \in InnerProduct\ \ V$
$\Leftrightarrow let\ \ x *_V\ y = Scale_V\ \ x\ \ y\ \ V$
$\quad and\ \$+_V = (\lambda x\ y \bullet\ Plus_V\ \ x\ \ y\ \ V)\ in$
$\quad (\forall\ a\ b\ p\ q\ r \bullet\ (a *_V\ p +_V\ b *_V\ q)\ \cdot_i\ r = (a *_V\ p)\ \cdot_i\ r +_R (b *_V\ q)\ \cdot_i\ r)$
$\wedge\ (\forall\ p\ q \bullet\ p\ \cdot_i\ q = q\ \cdot_i\ p)$
$\wedge\ (\forall\ p{:}'a \bullet\ p\ \cdot_i\ p \geqslant 0_R)$
$\wedge\ (\forall\ p{:}'a \bullet\ p\ \cdot_i\ p = 0_R \Rightarrow p = 0_V\ \ V)$

5.2 Inner Product Spaces

HOL Labelled Product

IPS

Rvs_{IPS}	$:'a\ RVS;$
Ip_{IPS}	$:'a \to 'a \to \mathbb{R}$

HOL Constant

$\boldsymbol{Ips} : {}'a\ IPS\ SET$

$\forall\ i{:}'a\ IPS \bullet\ i \in Ips$
$\Leftrightarrow\ Rvs_{IPS}\ \ i \in VS_R \wedge (Ip_{IPS}\ \ i) \in InnerProduct\ (Rvs_{IPS}\ \ i)$

HOL Constant

$\boldsymbol{Plus_I} :\ 'a \to 'a \to 'a\ IPS \to 'a;$
$\boldsymbol{Minus_I} :\ 'a \to 'a\ IPS \to 'a;$
$\boldsymbol{Subtract_I} :\ 'a \to 'a \to 'a\ IPS \to 'a;$
$\boldsymbol{0_I} :\ 'a\ IPS \to 'a;$
$\boldsymbol{Scale_I} :\ \mathbb{R} \to 'a \to 'a\ IPS \to 'a;$
$\boldsymbol{Ip_I} :\ 'a \to 'a \to 'a\ IPS \to \mathbb{R};$
$\boldsymbol{Norm_I} :\ 'a \to 'a\ IPS \to \mathbb{R}$

$\forall\ i{:}'a\ IPS \bullet$
$\quad (\forall v\ \ w \bullet\ Plus_I\ \ v\ \ w\ \ i = Plus_V\ \ v\ \ w\ (Rvs_{IPS}\ \ i))$
$\wedge\ (\forall v \bullet\ Minus_I\ \ v\ \ i = Minus_V\ \ v\ (Rvs_{IPS}\ \ i))$
$\wedge\ (\forall v\ \ w \bullet\ Subtract_I\ \ v\ \ w\ \ i = Plus_V\ \ v\ (Minus_V\ \ w\ (Rvs_{IPS}\ \ i))\ (Rvs_{IPS}\ \ i))$
$\wedge\ 0_I\ \ i = 0_V\ (Rvs_{IPS}\ \ i)$
$\wedge\ (\forall x\ \ v \bullet Scale_I\ \ x\ \ v\ \ i = Scale_V\ \ x\ \ v\ (Rvs_{IPS}\ \ i))$
$\wedge\ (\forall v\ \ w \bullet Ip_I\ \ v\ \ w\ \ i = Ip_{IPS}\ \ i\ \ v\ \ w)$
$\wedge\ (\forall v \bullet Norm_I\ \ v\ \ i = SqrtA(Ip_{IPS}\ \ i\ \ v\ \ v))$

6 MANIFOLDS

In order to reason about space-time and the nature of the physical universe it is necessary to have mathematical structures which abstract away from the specific coordinate systems of pre-relativistic physics.

My objective is to use formalisation as way of analysing the various arguments which are presented to justify moving from Aristotelian to Gallilean to special and general relativistic models of space-time.

According to [1] a manifold is a collection of coordinate patches over a topological space. This however, they recognise contains too much specific information about coordinate systems, An atlas of coordinate patches is necessary in defining the concept of differentiability, but is not itself a part of the manifold.

We therefore approach the definition of manifolds as the definition of certain properties of topologies, where a topology is itself just a set of sets satisfying certain properties.

6.1 Topology for Manifolds

A topological manifold is a topology which is:

- locally homeomorphic to $\mathbb{R}^n$ for some positive integer n

Sometimes it is required to be hausdorff, connected or paracompact but I don't propose to include any of these properties in the definition of a topological manifold. Presumably, if it is locally homeomorphic to $\mathbb{R}^n$ it will be hausdorff.

The concept "locally homeomorphic" can be defined without specific reference to $\mathbb{R}^n$, as a relationship between two topologies, or as here, by defining for any topology V the equivalence class of topologies with which it is locally homemorphic.

HOL Constant

$LocallyHomeomorphicTo : {'a}\ SET\ SET \to {'b}\ SET\ SET\ SET$

$\forall U\ V \bullet\ U \in LocallyHomeomorphicTo\ V \Leftrightarrow$
$\forall x \bullet\ x \in Space_T\ U \Rightarrow \exists y\ z\ f \bullet\ x \in y\ \land\ y \in U\ \land\ z \in V$
$\quad \land\ f \in (y \lhd_T U,\ z \lhd_T V)\ Homeomorphism$

A topological manifold:

HOL Constant

$TopologicalManifold : {'a}\ SET\ SET \to {'b}\ SET\ SET\ SET$

$\forall U\ V \bullet\ U \in TopologicalManifold\ V \Leftrightarrow$
$\quad U \in Topology\ \land\ V \in Topology\ \land\ U \in LocallyHomeomorphicTo\ V$

Normally the topology V will be the usual topology on $\mathbb{R}^n$ for some n.

There are two further steps in the definition of the kind of manifold we need. The first is the requirement of differentiability. It turns out that differentiability is a property not just of the topological manifold, but rather of the topology together with some system of coordinate patches. Such a collection of coordinate patches is called an atlas, and an atlas is differentiable if translations between the different coordinate systems it encompasses are all differentiable.

In order to define differentiable manifolds we seem to need to introduce coordinate systems, even though our definition of differentiability above is coordinate free. However, because we have a coordinate free definition of differentiability I can indulge in a modest (but probably not actually useful) “generalisation”.

Though it seems normal to expect the coordinate space to be in some power of the positive real line, its easier at this stage to allow an arbitrary normed vector space.

Differentiability is asserted using an atlas of coordinate patches. In this presentation a coordinate patch is an open set in some vector space.

HOL Labelled Product

ATLAS

Nvs_M $:'b\ NVS;$

$Cmap_M$ $: ('a\ SET \times ('a \to 'b))\ SET$

7 GEOMETRIC ALGEBRA

Geometric algebras may be the best way to do differential geometry. This section is a first sally in that direction.

There are two different strategies for exploiting geometric algebra and geometric analysis. One way is to construct a single geometric algebra which is the union of all the GA(n) for each finite n. Let us call this the concrete approach. This may be thought of by analogy with the treatment of real arithmetic in HOL and in contrast with the treament of algebraic theories such as that of groups or vector spaces.

The second approach is to follow the model provided by the treatement of groups and vector spaces.

The main factor determining which of these two approaches to follow is whether the structure in question is unique. Thus, in the case of the real numbers, the axioms are categorical, and so there is no obvious advantage from reasoning about ”real number structures” in general. In HOL there would be disadvantages, since such reasoning would be hypothetical. The theorems involve hypotheses about certain variables denoting the relevant kind of algebraic structure, and that certain other variables lie in the domain of the structures. These are unnecessary when reasoning about a single structure whose domain is a type.

It seems likely that the development of differential algebra in the context of an infinite dimensional geometric algebra would be the most efficient way to formalise physics. Its not quite so obvious that this would be good for the kinds of metaphysics which I have in mind as applications, but Rob Arthan is working with $GA(\infty, \infty)$, so my best bet is to build on his work.

Since my objective is to do metaphysics, and for the kind of metaphysics I have in mind, doing physics is a prerequisite, I propose pro-tem to follow Hestenes and try to formalise aspects of his accounts of physics using geometric algebra.

8 The Theory diffgeom

8.1 Parents

geomalg *rbjmisc*

8.2 Constants

$\boldsymbol{Scale_{RVS}}$	$'a\ RVS \to \mathbb{R} \to 'a \to 'a$
$\boldsymbol{Group_{RVS}}$	$'a\ RVS \to 'a\ GROUP$
$\boldsymbol{MkRVS}$	$'a\ GROUP \to (\mathbb{R} \to 'a \to 'a) \to 'a\ RVS$
$\boldsymbol{Scale_V}$	$\mathbb{R} \to 'a \to 'a\ RVS \to 'a$
$\boldsymbol{0_V}$	$'a\ RVS \to 'a$
$\boldsymbol{Subtract_V}$	$'a \to 'a \to 'a\ RVS \to 'a$
$\boldsymbol{Minus_V}$	$'a \to 'a\ RVS \to 'a$
$\boldsymbol{Plus_V}$	$'a \to 'a \to 'a\ RVS \to 'a$
$\boldsymbol{VS_R}$	$'a\ RVS\ \mathbb{P}$
$\mathbb{R}_{\boldsymbol{RVS}}$	$\mathbb{R}\ RVS$
$\boldsymbol{VectorSpaceProduct}$	
	$'a\ RVS \to 'b\ RVS \to ('a \times 'b)\ RVS$
$\boldsymbol{Lin}$	$'a\ RVS \times 'b\ RVS \to ('a \to 'b)\ \mathbb{P}$
$\boldsymbol{Fun_G}$	$'a\ NORM\ GROUP$
$\boldsymbol{Fun_{RVS}}$	$'a\ NORM\ RVS$
$\mathbb{R}^{\mathbf{2}}{}_{\boldsymbol{RVS}}$	$\mathbb{C}\ RVS$
$\mathbb{R}^{\mathbf{3}}{}_{\boldsymbol{RVS}}$	$\mathbb{R}^3\ RVS$
$\boldsymbol{0_{R3}}$	$\mathbb{R}^3$
$\$-_{\boldsymbol{tr}}$	$\mathbb{R}^3 \to \mathbb{R}^3 \to \mathbb{R}^3$
$\$+_{\boldsymbol{tr}}$	$\mathbb{R}^3 \to \mathbb{R}^3 \to \mathbb{R}^3$
$\$*_{\boldsymbol{trs}}$	$\mathbb{R}^3 \to \mathbb{R} \to \mathbb{R}^3$
$\$/_{\boldsymbol{trs}}$	$\mathbb{R}^3 \to \mathbb{R} \to \mathbb{R}^3$
$\boldsymbol{Norm}$	$'a\ RVS \to 'a\ NORM\ \mathbb{P}$
$\boldsymbol{NormProduct}$	$'a\ NORM \to 'b\ NORM \to ('a \times 'b)\ NORM$
$\boldsymbol{Di_{R3}}$	$\mathbb{R}^3\ NORM$
$\boldsymbol{Di_{R2}}$	$\mathbb{C}\ NORM$
$\boldsymbol{Di_R}$	$\mathbb{R}\ NORM$
$\boldsymbol{Norm_{NVS}}$	$'a\ NVS \to 'a\ NORM$
$\boldsymbol{Rvs_{NVS}}$	$'a\ NVS \to 'a\ RVS$
$\boldsymbol{MkNVS}$	$'a\ RVS \to 'a\ NORM \to 'a\ NVS$
$\boldsymbol{Nvs}$	$'a\ NVS\ \mathbb{P}$
$\boldsymbol{Norm_N}$	$'a \to 'a\ NVS\ NORM$
$\boldsymbol{Scale_N}$	$\mathbb{R} \to 'a \to 'a\ NVS \to 'a$
$\boldsymbol{0_N}$	$'a\ NVS \to 'a$
$\boldsymbol{Subtract_N}$	$'a \to 'a \to 'a\ NVS \to 'a$
$\boldsymbol{Minus_N}$	$'a \to 'a\ NVS \to 'a$
$\boldsymbol{Plus_N}$	$'a \to 'a \to 'a\ NVS \to 'a$
$\boldsymbol{NvsProduct}$	$'a\ NVS \to 'b\ NVS \to ('a \times 'b)\ NVS$
$\mathbb{R}^{\mathbf{4}}{}_{\boldsymbol{NVS}}$	$(\mathbb{R} \times \mathbb{R}^3)\ NVS$
$\mathbb{R}^{\mathbf{3}}{}_{\boldsymbol{NVS}}$	$\mathbb{R}^3\ NVS$
$\mathbb{R}^{\mathbf{2}}{}_{\boldsymbol{NVS}}$	$\mathbb{C}\ NVS$
$\mathbb{R}_{\boldsymbol{NVS}}$	$\mathbb{R}\ NVS$
$\boldsymbol{FrechetDeriv}$	$'a\ NVS \times 'b\ NVS \to ('a \to 'b) \to 'a \to ('a \to 'b)\ \mathbb{P}$

FDifferentiable	
	$'a\ NVS \times 'b\ NVS \rightarrow ('a \rightarrow 'b) \rightarrow 'a \rightarrow BOOL$
VDeriv	$'a\ NVS \rightarrow (\mathbb{R} \rightarrow 'a) \rightarrow \mathbb{R} \rightarrow 'a$
VNthDeriv	$\mathbb{N} \rightarrow 'b\ NVS \rightarrow (\mathbb{R} \rightarrow 'b) \rightarrow \mathbb{R} \rightarrow 'b$
EDiff	$'a\ NVS \times 'b\ NVS \rightarrow ('a \rightarrow 'b) \rightarrow ('a \rightarrow 'a \rightarrow 'b) \rightarrow BOOL$
NVSTopology	$'a\ NVS \rightarrow 'a\ \mathbb{P}\ \mathbb{P}$
InnerProduct	$'a\ RVS \rightarrow 'a\ IP\ \mathbb{P}$
$\boldsymbol{Ip_{IPS}}$	$'a\ IPS \rightarrow 'a\ IP$
$\boldsymbol{Rvs_{IPS}}$	$'a\ IPS \rightarrow 'a\ RVS$
MkIPS	$'a\ RVS \rightarrow 'a\ IP \rightarrow 'a\ IPS$
Ips	$'a\ IPS\ \mathbb{P}$
$\boldsymbol{Norm_I}$	$'a \rightarrow 'a\ IPS\ NORM$
$\boldsymbol{Ip_I}$	$'a \rightarrow 'a \rightarrow 'a\ IPS\ NORM$
$\boldsymbol{Scale_I}$	$\mathbb{R} \rightarrow 'a \rightarrow 'a\ IPS \rightarrow 'a$
$\boldsymbol{0_I}$	$'a\ IPS \rightarrow 'a$
$\boldsymbol{Subtract_I}$	$'a \rightarrow 'a \rightarrow 'a\ IPS \rightarrow 'a$
$\boldsymbol{Minus_I}$	$'a \rightarrow 'a\ IPS \rightarrow 'a$
$\boldsymbol{Plus_I}$	$'a \rightarrow 'a \rightarrow 'a\ IPS \rightarrow 'a$
LocallyHomeomorphicTo	
	$'a\ \mathbb{P}\ \mathbb{P} \rightarrow 'b\ \mathbb{P}\ \mathbb{P}\ \mathbb{P}$
TopologicalManifold	
	$'a\ \mathbb{P}\ \mathbb{P} \rightarrow 'b\ \mathbb{P}\ \mathbb{P}\ \mathbb{P}$
$\boldsymbol{Cmap_M}$	$('a, 'b)\ ATLAS \rightarrow 'a\ \mathbb{P} \leftrightarrow ('a \rightarrow 'b)$
$\boldsymbol{Nvs_M}$	$('a, 'b)\ ATLAS \rightarrow 'b\ NVS$
MkATLAS	$'b\ NVS \rightarrow 'a\ \mathbb{P} \leftrightarrow ('a \rightarrow 'b) \rightarrow ('a, 'b)\ ATLAS$

8.3 Aliases

Grp	$Group_{RVS} : 'a\ RVS \rightarrow 'a\ GROUP$
Scale	$Scale_{RVS} : 'a\ RVS \rightarrow \mathbb{R} \rightarrow 'a \rightarrow 'a$
+	$Plus_V : 'a \rightarrow 'a \rightarrow 'a\ RVS \rightarrow 'a$
~	$Minus_V : 'a \rightarrow 'a\ RVS \rightarrow 'a$
−	$Subtract_V : 'a \rightarrow 'a \rightarrow 'a\ RVS \rightarrow 'a$
$*_s$	$Scale_V : \mathbb{R} \rightarrow 'a \rightarrow 'a\ RVS \rightarrow 'a$
*	$VectorSpaceProduct : 'a\ RVS \rightarrow 'b\ RVS \rightarrow ('a \times 'b)\ RVS$
−	$\$-_{tr} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^3$
+	$\$-_{tr} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \rightarrow \mathbb{R}^3$
*	$\$*_{trs} : \mathbb{R}^3 \rightarrow \mathbb{R} \rightarrow \mathbb{R}^3$
/	$\$/_{trs} : \mathbb{R}^3 \rightarrow \mathbb{R} \rightarrow \mathbb{R}^3$
*	$NormProduct : 'a\ NORM \rightarrow 'b\ NORM \rightarrow ('a \times 'b)\ NORM$
+	$Plus_N : 'a \rightarrow 'a \rightarrow 'a\ NVS \rightarrow 'a$
~	$Minus_N : 'a \rightarrow 'a\ NVS \rightarrow 'a$
−	$Subtract_N : 'a \rightarrow 'a \rightarrow 'a\ NVS \rightarrow 'a$
*	$Scale_N : \mathbb{R} \rightarrow 'a \rightarrow 'a\ NVS \rightarrow 'a$
*	$NvsProduct : 'a\ NVS \rightarrow 'b\ NVS \rightarrow ('a \times 'b)\ NVS$

8.4 Types

'1 ***RVS***
'1 ***NVS***
'1 ***IPS***

$('1, '2)$ ***ATLAS***

8.5 Type Abbreviations

$\mathbb{R}^{\mathbf{3}}$	$\mathbb{R}^{3}$
$'a$ ***NORM***	$'a\ NORM$
$'a$ ***IP***	$'a\ IP$

8.6 Fixity

Right Infix 300:

$+_{tr}\ +_{V}\ -_{tr}$

Right Infix 310:

$*_{s}\quad *_{trs}\quad *_{V}\quad \cdot_{i}\quad /_{trs}$

8.7 Definitions

RVS $\vdash \exists\ f\bullet\ TypeDefn\ (\lambda\ x\bullet\ T)\ f$

MkRVS

Group$_{RVS}$

Scale$_{RVS}$ $\vdash \forall\ t\ x1\ x2$

$\bullet\ Grp\ (MkRVS\ x1\ x2) = x1$
$\wedge\ Scale\ (MkRVS\ x1\ x2) = x2$
$\wedge\ MkRVS\ (Grp\ t)\ (Scale\ t) = t$

Plus$_{V}$

Minus$_{V}$

Subtract$_{V}$

0_{V}

Scale$_{V}$ $\vdash \forall\ R$

$\bullet\ (\forall\ v\ w\bullet\ (v + w)\ R = (v\ .\ w)\ (Grp\ R))$
$\wedge\ (\forall\ v\bullet \sim v\ R = (v\ ^{\sim})\ (Grp\ R))$
$\wedge\ (\forall\ v\ w$
$\bullet\ (v - w)\ R = (v\ .\ (w\ ^{\sim})\ (Grp\ R))\ (Grp\ R))$
$\wedge\ 0_{V}\ R = Unit\ (Grp\ R)$
$\wedge\ (\forall\ x\ v\bullet\ (x *_{s} v)\ R = Scale\ R\ x\ v)$

VS$_{R}$ $\vdash \forall\ V$

$\bullet\ V \in VS_{R}$
$\Leftrightarrow Grp\ V \in AbelianGroup$
$\wedge\ Car\ (Grp\ V) = Universe$
$\wedge\ (\forall\ x\ v\ w$
$\bullet\ ((x *_{s} v)\ V + (x *_{s} w)\ V)\ V$
$= (x *_{s} (v + w)\ V)\ V)$
$\wedge\ (\forall\ x\ y\ v$
$\bullet\ ((x *_{s} v)\ V + (y *_{s} v)\ V)\ V$
$= ((x + y) *_{s} v)\ V)$
$\wedge\ (\forall\ x\ y\ v$
$\bullet\ (x *_{s} (y *_{s} v)\ V)\ V = ((x * y) *_{s} v)\ V)$
$\wedge\ (\forall\ v\bullet\ (1. *_{s} v)\ V = v)$

$\mathbb{R}_{\boldsymbol{RVS}}$ $\vdash \mathbb{R}_{RVS} = MkRVS\ \mathbb{R}_{+}\ (\lambda\ x\ y\bullet\ x * y)$

VectorSpaceProduct

$\vdash \forall\ V\ W$
$\quad \bullet\ V * W$
$\qquad = (let\ group = Grp\ V * Grp\ W$
$\qquad\quad and\ action\ r\ (ga,\ gb)$
$\qquad\qquad = ((r *_s ga)\ V,\ (r *_s gb)\ W)$
$\qquad\quad in\ MkRVS\ group\ action)$

Lin
$\vdash \forall\ V\ W\ f$
$\quad \bullet\ f \in Lin\ (V,\ W)$
$\qquad \Leftrightarrow f \in Homomorphism\ (Grp\ V,\ Grp\ W)$
$\qquad\quad \wedge\ (\forall\ x\ v\bullet\ f\ ((x *_s v)\ V) = (x *_s f\ v)\ W)$

Fun_G
$\vdash Fun_G$
$\quad = MkGROUP$
$\qquad Universe$
$\qquad (\lambda\ f\ g\ a\bullet\ f\ a + g\ a)$
$\qquad (\lambda\ a\bullet\ 0.)$
$\qquad (\lambda\ f\ a\bullet\ \sim (f\ a))$

Fun_{RVS}
$\vdash Fun_{RVS} = MkRVS\ Fun_G\ (\lambda\ x\ f\ a\bullet\ x * f\ a)$

$\mathbb{R}^2{}_{RVS}$
$\vdash \mathbb{R}^2{}_{RVS} = \mathbb{R}_{RVS} * \mathbb{R}_{RVS}$

$\mathbb{R}^3{}_{RVS}$
$\vdash \mathbb{R}^3{}_{RVS} = \mathbb{R}_{RVS} * \mathbb{R}_{RVS} * \mathbb{R}_{RVS}$

0_{R3}
$\vdash 0_{R3} = (0_R,\ 0_R,\ 0_R)$

$-_{tr}$
$\vdash \forall\ x_1\ y_1\ z_1\ x_2\ y_2\ z_2$
$\quad \bullet\ (x_1,\ y_1,\ z_1) + (x_2,\ y_2,\ z_2)$
$\qquad = (x_1 - x_2,\ y_1 - y_2,\ z_1 - z_2)$

$+_{tr}$
$\vdash \forall\ x_1\ y_1\ z_1\ x_2\ y_2\ z_2$
$\quad \bullet\ (x_1,\ y_1,\ z_1) +_{tr} (x_2,\ y_2,\ z_2)$
$\qquad = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$

$*_{trs}$
$\vdash \forall\ x_1\ y_1\ z_1\ r$
$\quad \bullet\ (x_1,\ y_1,\ z_1) * r = (x_1 * r,\ y_1 * r,\ z_1 * r)$

$/_{trs}$
$\vdash \forall\ x_1\ y_1\ z_1\ r$
$\quad \bullet\ (x_1,\ y_1,\ z_1) / r = (x_1 / r,\ y_1 / r,\ z_1 / r)$

Norm
$\vdash \forall\ V\ norm$
$\quad \bullet\ norm \in Norm\ V$
$\qquad \Leftrightarrow (\forall\ v\bullet\ 0. \leqslant norm\ v)$
$\qquad\quad \wedge\ (\forall\ v\bullet\ norm\ v = 0. \Leftrightarrow v = 0_V\ V)$
$\qquad\quad \wedge\ (\forall\ x\ v\bullet\ norm\ ((x *_s v)\ V) = Abs\ x * norm\ v)$
$\qquad\quad \wedge\ (\forall\ v\ w\bullet\ norm\ ((v + w)\ V) \leqslant norm\ v + norm\ w)$

NormProduct
$\vdash \forall\ n\ m\ a\ b$
$\quad \bullet\ (n * m)\ (a,\ b) = Abs\ (SqrtA\ (n\ a\ \hat{}\ 2 + m\ b\ \hat{}\ 2))$

Di_R
Di_{R2}
Di_{R3}
$\vdash (\forall\ r\bullet\ Di_R\ r = Abs\ r)$
$\quad \wedge\ Di_{R2} = Di_R * Di_R$
$\quad \wedge\ Di_{R3} = Di_R * Di_{R2}$

NVS
$\vdash \exists\ f\bullet\ TypeDefn\ (\lambda\ x\bullet\ T)\ f$

MkNVS
Rvs_{NVS}
$Norm_{NVS}$
$\vdash \forall\ t\ x1\ x2$
$\quad \bullet\ Rvs_{NVS}\ (MkNVS\ x1\ x2) = x1$
$\qquad \wedge\ Norm_{NVS}\ (MkNVS\ x1\ x2) = x2$
$\qquad \wedge\ MkNVS\ (Rvs_{NVS}\ t)\ (Norm_{NVS}\ t) = t$

Nvs
$\vdash \forall\ N$

$\bullet\ N \in Nvs$
$\Leftrightarrow Rvs_{NVS}\ N \in VS_R$
$\wedge\ Norm_{NVS}\ N \in Norm\ (Rvs_{NVS}\ N)$

Plus$_N$
Minus$_N$
Subtract$_N$
0_N
Scale$_N$
Norm$_N$ $\vdash \forall\ N$
$\bullet\ (\forall\ v\ w \bullet\ (v + w)\ N = (v + w)\ (Rvs_{NVS}\ N))$
$\wedge\ (\forall\ v \bullet \sim v\ N = \sim v\ (Rvs_{NVS}\ N))$
$\wedge\ (\forall\ v\ w$
$\bullet\ (v - w)\ N = (v + \sim w\ (Rvs_{NVS}\ N))\ (Rvs_{NVS}\ N))$
$\wedge\ 0_N\ N = 0_V\ (Rvs_{NVS}\ N)$
$\wedge\ (\forall\ x\ v \bullet\ (x * v)\ N = (x *_s v)\ (Rvs_{NVS}\ N))$
$\wedge\ (\forall\ v \bullet\ Norm_N\ v\ N = Norm_{NVS}\ N\ v)$

NvsProduct $\vdash \forall\ n\ m$
$\bullet\ n * m$
$= MkNVS$
$(Rvs_{NVS}\ n * Rvs_{NVS}\ m)$
$(Norm_{NVS}\ n * Norm_{NVS}\ m)$

$\mathbb{R}_{NVS}$
$\mathbb{R}^2{}_{NVS}$
$\mathbb{R}^3{}_{NVS}$
$\mathbb{R}^4{}_{NVS}$ $\vdash \mathbb{R}_{NVS} = MkNVS\ \mathbb{R}_{RVS}\ Di_R$
$\wedge\ \mathbb{R}^2{}_{NVS} = \mathbb{R}_{NVS} * \mathbb{R}_{NVS}$
$\wedge\ \mathbb{R}^3{}_{NVS} = \mathbb{R}_{NVS} * \mathbb{R}^2{}_{NVS}$
$\wedge\ \mathbb{R}^4{}_{NVS} = \mathbb{R}_{NVS} * \mathbb{R}^3{}_{NVS}$

FrechetDeriv $\vdash \forall\ M\ N\ f\ v\ D$
$\bullet\ D \in FrechetDeriv\ (M,\ N)\ f\ v$
$\Leftrightarrow D \in Lin\ (Rvs_{NVS}\ M,\ Rvs_{NVS}\ N)$
$\wedge\ (\forall\ e$
$\bullet\ 0. < e$
$\Rightarrow (\exists\ d$
$\bullet\ 0. < d$
$\wedge\ (\forall\ h$
$\bullet\ 0. < Norm_N\ h\ M \wedge Norm_N\ h\ M < d$
$\Rightarrow Norm_N$
$((Norm_N\ h\ M\ ^{-1}$
$*\ ((f\ ((v + h)\ M) - f\ v)$
N
$-\ D\ h)$
$N)$
$N)$
N
$< e)))$

FDifferentiable
$\vdash \forall\ M\ N\ f\ v$
$\bullet\ FDifferentiable\ (M,\ N)\ f\ v$
$\Leftrightarrow \neg\ FrechetDeriv\ (M,\ N)\ f\ v = \{\}$

VDeriv $\vdash \forall\ N\ f\ r$

$\bullet\ (let\ D = FrechetDeriv\ (\mathbb{R}_{NVS},\ N)\ f\ r$
$\quad in\ \neg\ D = \{\} \Rightarrow (\lambda\ r'\bullet\ (r' * VDeriv\ N\ f\ r)\ N) \in D)$

VNthDeriv $\vdash \forall\ n\ N\ f$
$\bullet\ VNthDeriv\ 0\ N\ f = f$
$\quad \wedge\ VNthDeriv\ (n + 1)\ N\ f$
$\quad\quad = (let\ f' = VDeriv\ N\ f\ in\ VNthDeriv\ n\ N\ f')$

EDiff $\vdash \forall\ N\ M\ f\ df$
$\bullet\ EDiff\ (N,\ M)\ f\ df$
$\quad \Leftrightarrow (\forall\ v\bullet\ df\ v \in FrechetDeriv\ (N,\ M)\ f\ v)$

NVSTopology $\vdash \forall\ v$
$\bullet\ NVSTopology\ v$
$\quad = \{vs$
$\quad |\forall\ x$
$\quad\quad \bullet\ x \in vs$
$\quad\quad\quad \Rightarrow (\exists\ \xi$
$\quad\quad\quad \bullet\ \forall\ y\bullet\ Norm_N\ ((y - x)\ v)\ v < \xi \Rightarrow y \in vs)\}$

InnerProduct $\vdash \forall\ V\ \$._i$
$\bullet\ \$._i \in InnerProduct\ V$
$\quad \Leftrightarrow (let\ x *_V y = (x *_s y)\ V$
$\quad\quad and\ x +_V y = (x + y)\ V$
$\quad\quad in\ (\forall\ a\ b\ p\ q\ r$
$\quad\quad\quad \bullet\ (a *_V p +_V b *_V q)\ ._i\ r$
$\quad\quad\quad\quad = (a *_V p)\ ._i\ r + (b *_V q)\ ._i\ r)$
$\quad\quad\quad \wedge\ (\forall\ p\ q\bullet\ p\ ._i\ q = q\ ._i\ p)$
$\quad\quad\quad \wedge\ (\forall\ p\bullet\ p\ ._i\ p \geqslant 0_R)$
$\quad\quad\quad \wedge\ (\forall\ p\bullet\ p\ ._i\ p = 0_R \Rightarrow p = 0_V\ V))$

IPS $\vdash \exists\ f\bullet\ TypeDefn\ (\lambda\ x\bullet\ T)\ f$
MkIPS
Rvs_{IPS}
Ip_{IPS} $\vdash \forall\ t\ x1\ x2$
$\bullet\ Rvs_{IPS}\ (MkIPS\ x1\ x2) = x1$
$\quad \wedge\ Ip_{IPS}\ (MkIPS\ x1\ x2) = x2$
$\quad \wedge\ MkIPS\ (Rvs_{IPS}\ t)\ (Ip_{IPS}\ t) = t$

Ips $\vdash \forall\ i$
$\bullet\ i \in Ips$
$\quad \Leftrightarrow Rvs_{IPS}\ i \in VS_R$
$\quad\quad \wedge\ Ip_{IPS}\ i \in InnerProduct\ (Rvs_{IPS}\ i)$

$Plus_I$
$Minus_I$
$Subtract_I$
0_I
$Scale_I$
Ip_I
$Norm_I$ $\vdash \forall\ i$
$\bullet\ (\forall\ v\ w\bullet\ Plus_I\ v\ w\ i = (v + w)\ (Rvs_{IPS}\ i))$
$\quad \wedge\ (\forall\ v\bullet\ Minus_I\ v\ i = \sim v\ (Rvs_{IPS}\ i))$
$\quad \wedge\ (\forall\ v\ w$
$\quad \bullet\ Subtract_I\ v\ w\ i$
$\quad\quad = (v + \sim w\ (Rvs_{IPS}\ i))\ (Rvs_{IPS}\ i))$
$\quad \wedge\ 0_I\ i = 0_V\ (Rvs_{IPS}\ i)$
$\quad \wedge\ (\forall\ x\ v\bullet\ Scale_I\ x\ v\ i = (x *_s v)\ (Rvs_{IPS}\ i))$

$\land\ (\forall\ v\ w\bullet\ Ip_I\ v\ w\ i = Ip_{IPS}\ i\ v\ w)$
$\land\ (\forall\ v\bullet\ Norm_I\ v\ i = SqrtA\ (Ip_{IPS}\ i\ v\ v))$

LocallyHomeomorphicTo

$\vdash \forall\ U\ V$
$\bullet\ U \in LocallyHomeomorphicTo\ V$
$\Leftrightarrow (\forall\ x$
$\bullet\ x \in Space_T\ U$
$\Rightarrow (\exists\ y\ z\ f$
$\bullet\ x \in y$
$\land\ y \in U$
$\land\ z \in V$
$\land\ f \in (y \lhd_T U,\ z \lhd_T V)\ Homeomorphism))$

TopologicalManifold

$\vdash \forall\ U\ V$
$\bullet\ U \in TopologicalManifold\ V$
$\Leftrightarrow U \in Topology$
$\land\ V \in Topology$
$\land\ U \in LocallyHomeomorphicTo\ V$

ATLAS $\vdash \exists\ f\bullet\ TypeDefn\ (\lambda\ x\bullet\ T)\ f$

MkATLAS

Nvs_M

$Cmap_M$ $\vdash \forall\ t\ x1\ x2$
$\bullet\ Nvs_M\ (MkATLAS\ x1\ x2) = x1$
$\land\ Cmap_M\ (MkATLAS\ x1\ x2) = x2$
$\land\ MkATLAS\ (Nvs_M\ t)\ (Cmap_M\ t) = t$

8.8 Theorems

schwartz_2nd_thm

$\vdash \forall\ u\ v\ x\ y\ A\ B\ C$
$\bullet\ A = u \mathbin{\hat{}} 2 + v \mathbin{\hat{}} 2$
$\land\ B = x \mathbin{\hat{}} 2 + y \mathbin{\hat{}} 2$
$\land\ C = u * x + v * y$
$\Rightarrow C \mathbin{\hat{}} 2 \leqslant A * B$

schwartz_2nd_thm1

$\vdash let\ (u,\ v)\ ._i\ (x,\ y) = u * x + v * y$
$in\ \forall\ u\ v\bullet\ (u\ ._i\ v) \mathbin{\hat{}} 2 \leqslant (u\ ._i\ u) * v\ ._i\ v$

schwartz_2nd_thm1b

$\vdash \forall\ \$._i$
$\bullet\ (\forall\ u\ v\ x\ y\bullet\ (u,\ v)\ ._i\ (x,\ y) = u * x + v * y)$
$\Rightarrow (\forall\ u\ v\bullet\ (u\ ._i\ v) \mathbin{\hat{}} 2 \leqslant (u\ ._i\ u) * v\ ._i\ v)$

ip_distrib_thm

$\vdash \forall\ \$._i$
$\bullet\ (\forall\ u\ v\ x\ y\bullet\ (u,\ v)\ ._i\ (x,\ y) = u * x + v * y)$
$\land\ (\forall\ u\ v\ x\ y$
$\bullet\ (u,\ v) +_V (x,\ y) = (u + x,\ v + y))$
$\Rightarrow (\forall\ x\ y$
$\bullet\ (x +_V y)\ ._i\ (x +_V y)$
$= x\ ._i\ x + 2. * x\ ._i\ y + y\ ._i\ y)$

schwartz_2nd_thm2

$\vdash let\ (u,\ v)\ ._i\ (x,\ y) = u * x + v * y$

$in\ let\ n\ v = SqrtA\ (v\ ._i\ v)$
$in\ \forall\ u\ v \bullet\ Abs\ (u\ ._i\ v) \leqslant n\ u * n\ v$

schwartz_2nd_thm2b

$$\vdash \forall\ \$._i\ n \bullet (\forall\ u\ v\ x\ y \bullet (u, v)\ ._i\ (x, y) = u * x + v * y) \land (\forall\ v \bullet\ n\ v = SqrtA\ (v\ ._i\ v)) \Rightarrow (\forall\ u\ v \bullet\ u\ ._i\ v \leqslant n\ u * n\ v)$$

triangle_ineq_thm

$$\vdash \forall\ \$._i\ \$+_V\ n \bullet (\forall\ u\ v\ x\ y \bullet (u, v)\ ._i\ (x, y) = u * x + v * y) \land (\forall\ u\ v\ x\ y \bullet (u, v) +_V (x, y) = (u + x, v + y)) \land (\forall\ v \bullet\ n\ v = SqrtA\ (v\ ._i\ v)) \Rightarrow (\forall\ u\ v \bullet\ n\ (u +_V v) \leqslant n\ u + n\ v)$$

triangle_ineq_thm2

$$\vdash let\ (u, v)\ ._i\ (x, y) = u * x + v * y\ and\ (u, v) +_V (x, y) = (u + x, v + y)\ in\ let\ n\ v = SqrtA\ (v\ ._i\ v)\ in\ \forall\ u\ v \bullet\ n\ (u +_V v) \leqslant n\ u + n\ v$$

vs_ops_def1

$$\vdash \forall\ R \bullet (\forall\ v\ w \bullet (v + w)\ R = (v\ .\ w)\ (Grp\ R)) \land (\forall\ v \bullet \sim v\ R = (v\ \tilde{})\ (Grp\ R)) \land (\forall\ v\ w \bullet (v - w)\ R = (v\ .\ (w\ \tilde{})\ (Grp\ R))\ (Grp\ R)) \land 0_V\ R = Unit\ (Grp\ R) \land (\forall\ x\ v \bullet (x *_s v)\ R = Scale\ R\ x\ v)$$

$\mathbb{R}_{RVS}$_VS_R_thm

$$\vdash \mathbb{R}_{RVS} \in VS_R$$

vector_product_thm

$$\vdash \forall\ V\ U \bullet\ V \in VS_R \land U \in VS_R \Rightarrow V * U \in VS_R$$

ℝ_Lin_thm1

$$\vdash \forall\ r \bullet\ \$*\ r \in Lin\ (\mathbb{R}_{RVS}, \mathbb{R}_{RVS})$$

ℝ_Lin_thm2

$$\vdash \forall\ V\ l \bullet\ V \in VS_R \Rightarrow (l \in Lin\ (\mathbb{R}_{RVS}, V) \Leftrightarrow l\ 1. \in Car\ (Grp\ V) \land (\forall\ r \bullet\ l\ r = (r *_s l\ 1.)\ V))$$

fun_g_group_thm

$$\vdash Fun_G \in Group$$

fun_v_vs_thm

$$\vdash Fun_{RVS} \in VS_R$$

ℝ_123_vs_thm

$$\vdash \mathbb{R}_{RVS} \in VS_R \land \mathbb{R}^2{}_{RVS} \in VS_R \land \mathbb{R}^3{}_{RVS} \in VS_R$$

NormProduct_thm

$$\vdash \forall\ V\ W\ n\ m \bullet\ V \in VS_R \land W \in VS_R \land n \in Norm\ V \land m \in Norm\ W \Rightarrow n * m \in Norm\ (V * W)$$

Di_R_Norm_thm

$$\vdash Di_R \in Norm\ \mathbb{R}_{RVS}$$

NvsProduct_thm

$$\vdash \forall\ N\ M \bullet\ N \in Nvs \land M \in Nvs \Rightarrow N * M \in Nvs$$

$\mathbb{R}_{NVS}$_Nvs_thm

$$\vdash \mathbb{R}_{NVS} \in Nvs \land \mathbb{R}^2{}_{NVS} \in Nvs$$

$\wedge \ \mathbb{R}^{3}{}_{NVS} \in Nvs$
$\wedge \ \mathbb{R}^{4}{}_{NVS} \in Nvs$

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