More Miscellanea (misc1, misc2)

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Abstract

This theory is for miscellanea which cannot be put in theory “rbjmisc” because of dependencies on other theories. It consists primarily of things required in the documents on non well-founded set theories, but not specific to that work, which make use of galactic set theory or fixed point theory. Since I moved my non-well-founded foundational work back from set theory to combinatory logic using version of well-founded set theory with urelements it has been necessary to replicate those definitions required which depend upon well-founded set theory in the context of this other version of well founded set theory. For that reason this document is in the process of being restructured as three theories, one of material which does not depend on the well founded set theory, and one for materials dependent respectively on each of the two versions of well-founded set theory. These are the theories misc1, misc2 and misc3.

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References

[1] Roger Bishop Jones. Miscellaneous Theory Supplements. *RBJones.com*, 2010. `http://www.rbjones.com/rbjpub/pp/doc/t006.pdf`.

1 INTRODUCTION

This was originally material which used to be in t024 and is moved here so that it can also be used in t026.

It should only contain material which cannot be put in "rbjmisc" because it depends on other theories, at this point GS or GSU and fixp. Some of it should therefore be moved up to "rbjmisc".

Three theories are created *misc*1, *misc*2, *misc*3.

*misc*2 depends on GS, *misc*3 on GSU. Both depend on *misc*1 which depends on neither version of set theory. Ultimately all that depends on neither GS nor GSU should migrate upwards into misc1, but this process has not yet been completed.

SML

```
open_theory "rbjmisc";
force_new_theory "misc1";
new_parent "fixp";
force_new_pc "'misc1";
merge_pcs ["'savedthm_cs_∃_proof"] "'misc1";
set_merge_pcs ["rbjmisc", "'misc1"];
```

2 DISCRETE PARTIAL ORDERS

SML

```
new_type_defn (["DPO"], "DPO", ["'a"],
    tac_proof (([], ⌜∃x:'a+BOOL• (λy•T) x⌝), ∃_tac ⌜InR T⌝ THEN prove_tac []));
```

HOL Constant

```
absDPO: 'a + BOOL → 'a DPO;
repDPO: 'a DPO → 'a + BOOL
─────────────────────────────
  (∀ a• absDPO (repDPO a) = a)
∧ (∀ r• repDPO (absDPO r) = r)
```

```
one_one_DPO_lemma =
    ⊢ OneOne repDPO ∧ OneOne absDPO
```

HOL Constant

```
dpoB : 'a DPO
─────────────────────────────
dpoB = absDPO(InR F)
```

HOL Constant

```
dpoT : 'a DPO
─────────────────────────────
dpoT = absDPO(InR T)
```

HOL Constant

$$\textbf{\textit{dpoE}} : {}'a \to {}'a\ DPO$$

$$\forall e \bullet\ dpoE\ e = absDPO(InL\ e)$$

HOL Constant

$$\textbf{\textit{dpoV}} : {}'a\ DPO \to {}'a$$

$$\forall x \bullet\ dpoV\ x = OutL\ (repDPO\ x)$$

HOL Constant

$$\textbf{\textit{dpoUdef}} : {}'a\ DPO \to BOOL$$

$$\forall x \bullet\ dpoUdef\ x \Leftrightarrow x = dpoB$$

HOL Constant

$$\textbf{\textit{dpoOdef}} : {}'a\ DPO \to BOOL$$

$$\forall x \bullet\ dpoOdef\ x \Leftrightarrow x = dpoT$$

$\textbf{\textit{dpo\_distinct\_clauses}} =$

$$\begin{aligned} \vdash\ & \neg\ dpoT = dpoB \\ & \wedge \neg\ dpoB = dpoT \\ & \wedge (\forall\ e \\ & \bullet \neg\ dpoE\ e = dpoT \\ & \quad \wedge \neg\ dpoE\ e = dpoB \\ & \quad \wedge \neg\ dpoT = dpoE\ e \\ & \quad \wedge \neg\ dpoB = dpoE\ e) \end{aligned}$$

$\textbf{\textit{dpoe\_inj\_thm}} =$

$$\vdash \forall\ e\ f \bullet (dpoE\ e = dpoE\ f) = e = f$$

$\textbf{\textit{dpoe\_inj\_lemma}} =$

$$\vdash \forall\ e\ f \bullet (dpoE\ e = dpoE\ f) = e = f$$

$\textbf{\textit{dpove\_lemma1}} =$

$$\vdash \forall\ e \bullet\ dpoV\ (dpoE\ e) = e$$

$\textbf{\textit{dpodef\_lemma1}} =$

$$\begin{aligned} \vdash\ & dpoUdef\ dpoB \\ & \wedge dpoOdef\ dpoT \\ & \wedge \neg\ dpoUdef\ dpoT \\ & \wedge \neg\ dpoOdef\ dpoB \\ & \wedge (\forall\ e \bullet \neg\ dpoUdef\ (dpoE\ e) \wedge \neg\ dpoOdef\ (dpoE\ e)) \end{aligned}$$

$\boldsymbol{dpo\_cases\_thm}$ =

$\vdash \forall\ x \bullet\ x = dpoB \lor x = dpoT \lor (\exists\ e \bullet\ x = dpoE\ e)$

$\boldsymbol{dpoev\_lemma1}$ =

$\vdash \forall\ x \bullet\ \neg\ dpoUdef\ x \land \neg\ dpoOdef\ x \Rightarrow dpoE\ (dpoV\ x) = x$

$\boldsymbol{dpo\_rpou\_lemma}$ =

$\vdash RpoU\ Dpo$

$\boldsymbol{dpo\_glbs\_exist\_thm}$ =

$\vdash GlbsExist\ Dpo$

$\boldsymbol{dpo\_lubs\_exist\_thm}$ =

$\vdash LubsExist\ Dpo$

HOL Constant

$\boldsymbol{Dpo} : {'a}\ DPO \to {'a}\ DPO \to BOOL$

$\forall x\ y \bullet\ Dpo\ x\ y \Leftrightarrow x = y \lor x = dpoB \lor y = dpoT$

3 TRUTH VALUES

I am uncertain at this point whether to work with three or four truth values, so both of these are provided for here.

3.1 Three Valued

SML

```
declare_type_abbrev("TTV", [], ⌜:BOOL OPT⌝);
```

HOL Constant

$\boldsymbol{pTrue} : TTV$

$pTrue = Value\ T$

HOL Constant

$\boldsymbol{pFalse} : TTV$

$pFalse = Value\ F$

HOL Constant

$\boldsymbol{pU} : TTV$

$pU = Undefined$

```
tv_cases_thm =
    ⊢ ∀ x• x = pTrue ∨ x = pFalse ∨ x = pU

tv_distinct_clauses =
    ⊢ ¬ pTrue = pFalse
        ∧ ¬ pTrue = pU
        ∧ ¬ pFalse = pTrue
        ∧ ¬ pFalse = pU
        ∧ ¬ pU = pTrue
        ∧ ¬ pU = pFalse
```

SML

```
declare_infix(300, "≤t3");
```

SML

```
declare_type_abbrev ("REL", ["'a"], ⌜:'a → 'a → BOOL⌝);
```

First an ordering on the "truth values" is defined.

HOL Constant

```
$≤t3 : TTV REL
──────────────────────────────
∀ t1 t2•
    t1 ≤t3 t2 ⇔ t1 = t2 ∨ t1 = pU
```

```
≤t3_refl_thm =
    ⊢ ∀ x• x ≤t3 x

≤t3_trans_thm =
    ⊢ ∀ x y z• x ≤t3 y ∧ y ≤t3 z ⇒ x ≤t3 z

≤t3_antisym_thm =
    ⊢ ∀ x y• x ≤t3 y ∧ y ≤t3 x ⇒ x = y

≤t3_partialorder_thm =
    ⊢ ∀ Y• PartialOrder (Y, $≤t3)

≤t3_clauses =
  ⊢ pU ≤t3 pTrue
    ∧ pU ≤t3 pFalse
    ∧ ¬ pTrue ≤t3 pU
    ∧ ¬ pFalse ≤t3 pU
    ∧ ¬ pFalse ≤t3 pTrue
    ∧ ¬ pTrue ≤t3 pFalse

lin_≤t3_lemma =
```

$\vdash \forall\ Y \bullet\ LinearOrder\ (Y,\ \$\leq_{t3}) \Leftrightarrow \neg\ pTrue \in Y \vee \neg\ pFalse \in Y$

$\boldsymbol{lin\_\leq_{t3}\_cases\_lemma} =$

$\vdash \forall\ Y$
$\quad \bullet\ LinearOrder\ (Y,\ \$\leq_{t3})$
$\qquad \Leftrightarrow Y = \{\}$
$\qquad\quad \vee\ Y = \{pU\}$
$\qquad\quad \vee\ Y = \{pTrue\}$
$\qquad\quad \vee\ Y = \{pFalse\}$
$\qquad\quad \vee\ Y = \{pU;\ pTrue\}$
$\qquad\quad \vee\ Y = \{pU;\ pFalse\}$

$\boldsymbol{\leq_{t3}\_isub\_cases\_lemma} =$

$\vdash \forall\ Y$
$\quad \bullet\ IsUb\ \$\leq_{t3}\ \{\} = (\lambda\ x \bullet\ T)$
$\qquad \wedge\ IsUb\ \$\leq_{t3}\ \{pU\} = (\lambda\ x \bullet\ T)$
$\qquad \wedge\ IsUb\ \$\leq_{t3}\ \{pTrue\} = (\lambda\ x \bullet\ x = pTrue)$
$\qquad \wedge\ IsUb\ \$\leq_{t3}\ \{pFalse\} = (\lambda\ x \bullet\ x = pFalse)$
$\qquad \wedge\ IsUb\ \$\leq_{t3}\ \{pU;\ pTrue\} = (\lambda\ x \bullet\ x = pTrue)$
$\qquad \wedge\ IsUb\ \$\leq_{t3}\ \{pU;\ pFalse\} = (\lambda\ x \bullet\ x = pFalse)$

$\boldsymbol{\leq_{t3}\_islub\_clauses} =$

$\vdash \forall\ Y$
$\quad \bullet\ IsLub\ \$\leq_{t3}\ \{\}\ pU$
$\qquad \wedge\ IsLub\ \$\leq_{t3}\ \{pU\}\ pU$
$\qquad \wedge\ IsLub\ \$\leq_{t3}\ \{pTrue\}\ pTrue$
$\qquad \wedge\ IsLub\ \$\leq_{t3}\ \{pFalse\}\ pFalse$
$\qquad \wedge\ IsLub\ \$\leq_{t3}\ \{pU;\ pTrue\}\ pTrue$
$\qquad \wedge\ IsLub\ \$\leq_{t3}\ \{pU;\ pFalse\}\ pFalse$

$\boldsymbol{chaincomplete\_\leq_{t3}\_kemma} =$

$\qquad \vdash ChainComplete\ (Universe,\ \$\leq_{t3})$

$\boldsymbol{ccrpou\_\leq_{t3}\_thm} =$

$\qquad \vdash CcRpoU\ \$\leq_{t3}$

3.2 Four Valued

SML

```
declare_type_abbrev("FTV", [], ⌜:BOOL DPO⌝);
```

HOL Constant

$\boldsymbol{fTrue} : FTV$

$fTrue = dpoE\ T$

HOL Constant

$\boldsymbol{fFalse} : FTV$

$fFalse = dpoE\ F$

HOL Constant

$\boldsymbol{fB} : FTV$

$fB = dpoB$

HOL Constant

$\boldsymbol{fT} : FTV$

$fT = dpoT$

$\boldsymbol{ftv\_cases\_thm} =$
$\quad \vdash \forall\ x \bullet\ x = fTrue \lor x = fFalse \lor x = fB \lor x = fT$
$\boldsymbol{ftv\_distinct\_clauses} =$
$\quad \vdash \neg\ fTrue = fFalse$
$\quad \land \neg\ fTrue = fB$
$\quad \land \neg\ fTrue = fT$
$\quad \land \neg\ fFalse = fTrue$
$\quad \land \neg\ fFalse = fB$
$\quad \land \neg\ fFalse = fT$
$\quad \land \neg\ fB = fTrue$
$\quad \land \neg\ fB = fFalse$
$\quad \land \neg\ fB = fT$
$\quad \land \neg\ fT = fTrue$
$\quad \land \neg\ fT = fFalse$
$\quad \land \neg\ fT = fB$

$\boldsymbol{ftvs\_cases\_thm} =$
$\vdash \forall\ x$
$\bullet\ x = \{\}$
$\quad \lor x = \{fB\}$
$\quad \lor x = \{fFalse\}$
$\quad \lor x = \{fTrue\}$
$\quad \lor x = \{fT\}$
$\quad \lor x = \{fB;\ fFalse\}$
$\quad \lor x = \{fB;\ fTrue\}$
$\quad \lor x = \{fB;\ fT\}$
$\quad \lor x = \{fFalse;\ fTrue\}$
$\quad \lor x = \{fFalse;\ fT\}$
$\quad \lor x = \{fTrue;\ fT\}$

$\lor\ x = \{fB;\ fFalse;\ fTrue\}$
$\lor\ x = \{fB;\ fFalse;\ fT\}$
$\lor\ x = \{fB;\ fTrue;\ fT\}$
$\lor\ x = \{fFalse;\ fTrue;\ fT\}$
$\lor\ x = \{fB;\ fFalse;\ fTrue;\ fT\}$

SML

```
declare_infix(300, "≤t4");
```

Now an ordering on these truth values is defined.

HOL Constant

$\mathbf{\$\leq_{t4}} : FTV\ REL$

$\forall\ t1\ t2 \bullet$
$t1 \leq_{t4} t2 \Leftrightarrow t1 = t2 \lor t1 = fB \lor t2 = fT$

$\leq_{t4}$_**dpo_thm** =
$\vdash \$\leq_{t4} = Dpo$

$\leq_{t4}$_**refl_thm** =
$\vdash \forall\ x \bullet\ x \leq_{t4} x$

$\leq_{t4}$_**trans_thm** =
$\vdash \forall\ x\ y\ z \bullet\ x \leq_{t4} y \land y \leq_{t4} z \Rightarrow x \leq_{t4} z$

$\leq_{t4}$_**antisym_thm** =
$\vdash \forall\ x\ y \bullet\ x \leq_{t4} y \land y \leq_{t4} x \Rightarrow x = y$

$\leq_{t4}$_**antisym_thm2** =
$\vdash Antisym\ (Universe, \$\leq_{t4})$

ft_fb_thm =
$\vdash \forall\ x \bullet\ (fT \leq_{t4} x \Leftrightarrow x = fT) \land (x \leq_{t4} fB \Leftrightarrow x = fB)$

$\leq_{t4}$_**partialorder_thm** =
$\vdash \forall\ Y \bullet\ PartialOrder\ (Y, \$\leq_{t4})$

$\leq_{t4}$_**lin_lemma** =
$\vdash \forall\ Y \bullet\ LinearOrder\ (Y, \$\leq_{t4}) = (\neg\ fTrue \in Y \lor \neg\ fFalse \in Y)$

$\leq_{t4}$_**clauses** =
$\vdash (\forall\ x \bullet\ fB \leq_{t4} x)$
$\land (\forall\ x \bullet\ x \leq_{t4} fT)$
$\land \neg\ fTrue \leq_{t4} fB$
$\land \neg\ fFalse \leq_{t4} fB$

$\land \lnot fT \leq_{t4} fB$
$\land \lnot fFalse \leq_{t4} fTrue$
$\land \lnot fT \leq_{t4} fTrue$
$\land \lnot fTrue \leq_{t4} fFalse$
$\land \lnot fT \leq_{t4} fFalse$

$\boldsymbol{\leq_{t4}\_lin\_cases\_lemma} =$
$\vdash \forall\ Y$
$\bullet\ LinearOrder\ (Y, \$\leq_{t4})$
$= (Y = \{\}$
$\lor\ Y = \{fB\}$
$\lor\ Y = \{fTrue\}$
$\lor\ Y = \{fFalse\}$
$\lor\ Y = \{fT\}$
$\lor\ Y = \{fB; fTrue\}$
$\lor\ Y = \{fB; fFalse\}$
$\lor\ Y = \{fB; fT\}$
$\lor\ Y = \{fTrue; fT\}$
$\lor\ Y = \{fFalse; fT\}$
$\lor\ Y = \{fB; fTrue; fT\}$
$\lor\ Y = \{fB; fFalse; fT\})$

$\boldsymbol{gt\_false\_true\_lemma} =$
$\vdash \forall\ x \bullet\ fFalse \leq_{t4} x \land fTrue \leq_{t4} x \Rightarrow x = fT$

$\boldsymbol{lt\_false\_true\_lemma} =$
$\vdash \forall\ x \bullet\ x \leq_{t4} fFalse \land x \leq_{t4} fTrue \Rightarrow x = fB$

$\boldsymbol{gt\_ft\_lemma} =$
$\vdash \forall\ x \bullet\ fT \leq_{t4} x \Rightarrow x = fT$

$\boldsymbol{lt\_fb\_lemma} =$
$\vdash \forall\ x \bullet\ x \leq_{t4} fB \Rightarrow x = fB$

$\boldsymbol{sg\_ftrue\_lemma} =$
$\vdash \forall\ x \bullet\ fTrue \leq_{t4} x \land \lnot x = fTrue \Rightarrow x = fT$

$\boldsymbol{sl\_ftrue\_lemma} =$
$\vdash \forall\ x \bullet\ x \leq_{t4} fTrue \land \lnot x = fTrue \Rightarrow x = fB$

$\boldsymbol{sg\_ffalse\_lemma} =$
$\vdash \forall\ x \bullet\ fFalse \leq_{t4} x \land \lnot x = fFalse \Rightarrow x = fT$

$\boldsymbol{sl\_ffalse\_lemma} =$
$\vdash \forall\ x \bullet\ x \leq_{t4} fFalse \land \lnot x = fFalse \Rightarrow x = fB$

$\boldsymbol{eq\_ft\_fc\_clauses} =$

$\vdash \forall\ x$

$\bullet\ fFalse \leq_{t4}\ x \land \neg\ x = fFalse$
$\lor\ fTrue \leq_{t4}\ x \land \neg\ x = fTrue$
$\lor\ fT \leq_{t4}\ x$
$\lor\ fFalse \leq_{t4}\ x \land fTrue \leq_{t4}\ x$
$\Rightarrow x = fT$

$\boldsymbol{eq\_fb\_fc\_clauses} =$

$\vdash \forall\ x$

$\bullet\ x \leq_{t4}\ fFalse \land \neg\ x = fFalse$
$\lor\ x \leq_{t4}\ fTrue \land \neg\ x = fTrue$
$\lor\ x \leq_{t4}\ fB$
$\lor\ x \leq_{t4}\ fTrue \land x \leq_{t4}\ fFalse$
$\Rightarrow x = fB$

$\boldsymbol{\leq_{t4}\_isub\_clauses} =$

$\vdash (\forall x \bullet\ IsUb\ \{x\}\ x)$
$\land\ IsUb\ \$\leq_{t4}\ \{\} = (\lambda\ x \bullet\ T)$
$\land\ IsUb\ \$\leq_{t4}\ \{fB\} = (\lambda\ x \bullet\ T)$
$\land\ IsUb\ \$\leq_{t4}\ \{fTrue\} = (\lambda\ x \bullet\ x = fTrue \lor x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fFalse\} = (\lambda\ x \bullet\ x = fFalse \lor x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fB;\ fTrue\} = (\lambda\ x \bullet\ x = fTrue \lor x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fB;\ fFalse\} = (\lambda\ x \bullet\ x = fFalse \lor x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fFalse;\ fTrue\} = (\lambda\ x \bullet\ x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fB;\ fFalse;\ fTrue\} = (\lambda\ x \bullet\ x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fT\} = (\lambda\ x \bullet\ x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fB;\ fT\} = (\lambda\ x \bullet\ x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fFalse;\ fT\} = (\lambda\ x \bullet\ x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fTrue;\ fT\} = (\lambda\ x \bullet\ x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fB;\ fFalse;\ fT\} = (\lambda\ x \bullet\ x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fB;\ fTrue;\ fT\} = (\lambda\ x \bullet\ x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fB;\ fFalse;\ fTrue;\ fT\} = (\lambda\ x \bullet\ x = fT)$
$\land\ IsUb\ \$\leq_{t4}\ \{fFalse;\ fTrue;\ fT\} = (\lambda\ x \bullet\ x = fT)$

$\boldsymbol{\leq_{t4}\_islb\_clauses} =$

$\vdash (\forall\ x \bullet\ IsLb\ \$\leq_{t4}\ \{x\}\ x)$
$\land\ IsLb\ \$\leq_{t4}\ \{\} = (\lambda\ x \bullet\ T)$
$\land\ IsLb\ \$\leq_{t4}\ \{fB\} = (\lambda\ x \bullet\ x = fB)$
$\land\ IsLb\ \$\leq_{t4}\ \{fTrue\} = (\lambda\ x \bullet\ x = fTrue \lor x = fB)$
$\land\ IsLb\ \$\leq_{t4}\ \{fFalse\} = (\lambda\ x \bullet\ x = fFalse \lor x = fB)$
$\land\ IsLb\ \$\leq_{t4}\ \{fB;\ fTrue\} = (\lambda\ x \bullet\ x = fB)$
$\land\ IsLb\ \$\leq_{t4}\ \{fB;\ fFalse\} = (\lambda\ x \bullet\ x = fB)$
$\land\ IsLb\ \$\leq_{t4}\ \{fFalse;\ fTrue\} = (\lambda\ x \bullet\ x = fB)$
$\land\ IsLb\ \$\leq_{t4}\ \{fB;\ fFalse;\ fTrue\} = (\lambda\ x \bullet\ x = fB)$

$\wedge$ *IsLb* $\$\leq_{t4}$ $\{fT\} = (\lambda\ x\bullet\ T)$
$\wedge$ *IsLb* $\$\leq_{t4}$ $\{fB; fT\} = (\lambda\ x\bullet\ x = fB)$
$\wedge$ *IsLb* $\$\leq_{t4}$ $\{fFalse; fT\} = (\lambda\ x\bullet\ x = fFalse \vee x = fB)$
$\wedge$ *IsLb* $\$\leq_{t4}$ $\{fTrue; fT\} = (\lambda\ x\bullet\ x = fTrue \vee x = fB)$
$\wedge$ *IsLb* $\$\leq_{t4}$ $\{fB; fFalse; fT\} = (\lambda\ x\bullet\ x = fB)$
$\wedge$ *IsLb* $\$\leq_{t4}$ $\{fB; fTrue; fT\} = (\lambda\ x\bullet\ x = fB)$
$\wedge$ *IsLb* $\$\leq_{t4}$ $\{fB; fFalse; fTrue; fT\} = (\lambda\ x\bullet\ x = fB)$
$\wedge$ *IsLb* $\$\leq_{t4}$ $\{fFalse; fTrue; fT\} = (\lambda\ x\bullet\ x = fB)$

$\leq_{t4}$_islub_clauses =
$\vdash$ $(\forall x\bullet$ *IsLub* $\$\leq_{t4}$ $\{x\}\ x)$
$\wedge$ *IsLub* $\$\leq_{t4}$ $\{\}\ fB$
$\wedge$ *IsLub* $\$\leq_{t4}$ $\{fB; fTrue\}\ fTrue$
$\wedge$ *IsLub* $\$\leq_{t4}$ $\{fB; fFalse\}\ fFalse$
$\wedge$ *IsLub* $\$\leq_{t4}$ $\{fFalse; fTrue\}\ fT$
$\wedge$ *IsLub* $\$\leq_{t4}$ $\{fB; fFalse; fTrue\}\ fT$
$\wedge$ *IsLub* $\$\leq_{t4}$ $\{fB; fT\}\ fT$
$\wedge$ *IsLub* $\$\leq_{t4}$ $\{fB; fTrue; fT\}\ fT$
$\wedge$ *IsLub* $\$\leq_{t4}$ $\{fB; fFalse; fT\}\ fT$
$\wedge$ *IsLub* $\$\leq_{t4}$ $\{fB; fTrue; fFalse; fT\}\ fT$
$\wedge$ *IsLub* $\$\leq_{t4}$ $\{fTrue; fT\}\ fT$
$\wedge$ *IsLub* $\$\leq_{t4}$ $\{fFalse; fT\}\ fT$
$\wedge$ *IsLub* $\$\leq_{t4}$ $\{fTrue; fFalse; fT\}\ fT$

$\leq_{t4}$_isglb_clauses =
$\vdash$ $(\forall\ x\bullet$ *IsGlb* $\$\leq_{t4}$ $\{x\}\ x)$
$\wedge$ *IsGlb* $\$\leq_{t4}$ $\{\}\ fT$
$\wedge$ *IsGlb* $\$\leq_{t4}$ $\{fB; fFalse\}\ fB$
$\wedge$ *IsGlb* $\$\leq_{t4}$ $\{fB; fTrue\}\ fB$
$\wedge$ *IsGlb* $\$\leq_{t4}$ $\{fFalse; fTrue\}\ fB$
$\wedge$ *IsGlb* $\$\leq_{t4}$ $\{fB; fFalse; fTrue\}\ fB$
$\wedge$ *IsGlb* $\$\leq_{t4}$ $\{fB; fT\}\ fB$
$\wedge$ *IsGlb* $\$\leq_{t4}$ $\{fB; fFalse; fT\}\ fB$
$\wedge$ *IsGlb* $\$\leq_{t4}$ $\{fB; fTrue; fT\}\ fB$
$\wedge$ *IsGlb* $\$\leq_{t4}$ $\{fB; fFalse; fTrue; fT\}\ fB$
$\wedge$ *IsGlb* $\$\leq_{t4}$ $\{fFalse; fT\}\ fFalse$
$\wedge$ *IsGlb* $\$\leq_{t4}$ $\{fTrue; fT\}\ fTrue$
$\wedge$ *IsGlb* $\$\leq_{t4}$ $\{fFalse; fTrue; fT\}\ fB$

$\leq_{t4}$_lin_cases_lemma =
$\vdash \forall\ Y$
$\bullet$ *LinearOrder* $(Y, \$\leq_{t4})$
$\Leftrightarrow Y = \{\}$
$\vee\ Y = \{fB\}$
$\vee\ Y = \{fTrue\}$

$\lor\ Y = \{fFalse\}$
$\lor\ Y = \{fT\}$
$\lor\ Y = \{fB;\ fTrue\}$
$\lor\ Y = \{fB;\ fFalse\}$
$\lor\ Y = \{fB;\ fT\}$
$\lor\ Y = \{fTrue;\ fT\}$
$\lor\ Y = \{fFalse;\ fT\}$
$\lor\ Y = \{fB;\ fTrue;\ fT\}$
$\lor\ Y = \{fB;\ fFalse;\ fT\}$

$\leq_{t4}$***_glbs_exist_thm*** =
$\vdash\ GlbsExist\ \$\leq_{t4}$

$\leq_{t4}$***_lubs_exist_thm*** =
$\vdash\ LubsExist\ \$\leq_{t4}$

$\leq_{t4}$***_lub_islub_lemma*** =
$\vdash \forall\ s\ e \bullet\ Lub\ \$\leq_{t4}\ s = e \Leftrightarrow IsLub\ \$\leq_{t4}\ s\ e$

$\leq_{t4}$***_lub_clauses*** =
$\vdash\ Lub\ \$\leq_{t4}\ \{\} = fB$
$\land\ Lub\ \$\leq_{t4}\ \{fT\} = fT$
$\land\ Lub\ \$\leq_{t4}\ \{fTrue\} = fTrue$
$\land\ Lub\ \$\leq_{t4}\ \{fTrue;\ fT\} = fT$
$\land\ Lub\ \$\leq_{t4}\ \{fFalse\} = fFalse$
$\land\ Lub\ \$\leq_{t4}\ \{fFalse;\ fT\} = fT$
$\land\ Lub\ \$\leq_{t4}\ \{fFalse;\ fTrue\} = fT$
$\land\ Lub\ \$\leq_{t4}\ \{fFalse;\ fTrue;\ fT\} = fT$
$\land\ Lub\ \$\leq_{t4}\ \{fB\} = fB$
$\land\ Lub\ \$\leq_{t4}\ \{fB;\ fT\} = fT$
$\land\ Lub\ \$\leq_{t4}\ \{fB;\ fTrue\} = fTrue$
$\land\ Lub\ \$\leq_{t4}\ \{fB;\ fTrue;\ fT\} = fT$
$\land\ Lub\ \$\leq_{t4}\ \{fB;\ fFalse\} = fFalse$
$\land\ Lub\ \$\leq_{t4}\ \{fB;\ fFalse;\ fT\} = fT$
$\land\ Lub\ \$\leq_{t4}\ \{fB;\ fFalse;\ fTrue\} = fT$
$\land\ Lub\ \$\leq_{t4}\ \{fB;\ fFalse;\ fTrue;\ fT\} = fT$

$\leq_{t4}$***_lub_thm*** =
$\vdash \forall\ s$
$\bullet\ Lub\ \$\leq_{t4}\ s$
$= (if\ fT \in s$
$then\ fT$
$else\ if\ fTrue \in s$
$then\ if\ fFalse \in s\ then\ fT\ else\ fTrue$
$else\ if\ fFalse \in s$

then *fFalse*
else *fB*)

$\leq_{t4}$-***glb_clauses*** =

$\vdash Glb\ \$\leq_{t4}\ \{\} = fT$
$\wedge\ Glb\ \$\leq_{t4}\ \{fT\} = fT$
$\wedge\ Glb\ \$\leq_{t4}\ \{fTrue\} = fTrue$
$\wedge\ Glb\ \$\leq_{t4}\ \{fTrue;\ fT\} = fTrue$
$\wedge\ Glb\ \$\leq_{t4}\ \{fFalse\} = fFalse$
$\wedge\ Glb\ \$\leq_{t4}\ \{fFalse;\ fT\} = fFalse$
$\wedge\ Glb\ \$\leq_{t4}\ \{fFalse;\ fTrue\} = fB$
$\wedge\ Glb\ \$\leq_{t4}\ \{fFalse;\ fTrue;\ fT\} = fB$
$\wedge\ Glb\ \$\leq_{t4}\ \{fB\} = fB$
$\wedge\ Glb\ \$\leq_{t4}\ \{fB;\ fT\} = fB$
$\wedge\ Glb\ \$\leq_{t4}\ \{fB;\ fTrue\} = fB$
$\wedge\ Glb\ \$\leq_{t4}\ \{fB;\ fTrue;\ fT\} = fB$
$\wedge\ Glb\ \$\leq_{t4}\ \{fB;\ fFalse\} = fB$
$\wedge\ Glb\ \$\leq_{t4}\ \{fB;\ fFalse;\ fT\} = fB$
$\wedge\ Glb\ \$\leq_{t4}\ \{fB;\ fFalse;\ fTrue\} = fB$
$\wedge\ Glb\ \$\leq_{t4}\ \{fB;\ fFalse;\ fTrue;\ fT\}$

The following definitions expresses compatibility of truth values. This is useful when trying to obtain a boolean valued function as a total least fixed point of a monotonic four valued function.

I first define a notion of compatibility between truth values.

HOL Constant

CompFTV : *FTV SET SET*

$CompFTV = \{\{\};\ \{fB\};\ \{fFalse\};\ \{fTrue\};\ \{fB;\ fFalse\};\ \{fB;\ fTrue\}\}$

and its dual:

HOL Constant

CoCompFTV : *FTV SET SET*

$CoCompFTV = \{\{\};\ \{fT\};\ \{fFalse\};\ \{fTrue\};\ \{fFalse;\ fT\};\ \{fTrue;\ fT\}\}$

compftv_lemma =

$\vdash \forall\ s$
$\bullet\ s \in CompFTV \Leftrightarrow \neg\ fT \in s \wedge (\neg\ fTrue \in s \vee \neg\ fFalse \in s)$

cocompftv_lemma =

$\vdash \forall\ s$
$\bullet\ s \in CoCompFTV \Leftrightarrow \neg\ fB \in s \wedge (\neg\ fTrue \in s \vee \neg\ fFalse \in s)$

$\boldsymbol{CompFTV\_\in\_clauses} =$

$\vdash \{\} \in CompFTV$

$\quad \land \{fB\} \in CompFTV$

$\quad \land \{fFalse\} \in CompFTV$

$\quad \land \{fTrue\} \in CompFTV$

$\quad \land \neg \{fT\} \in CompFTV$

$\quad \land \{fB; fFalse\} \in CompFTV$

$\quad \land \{fB; fTrue\} \in CompFTV$

$\quad \land \neg \{fB; fT\} \in CompFTV$

$\quad \land \neg \{fFalse; fTrue\} \in CompFTV$

$\quad \land \neg \{fFalse; fT\} \in CompFTV$

$\quad \land \neg \{fTrue; fT\} \in CompFTV$

$\quad \land \neg \{fB; fFalse; fTrue\} \in CompFTV$

$\quad \land \neg \{fB; fFalse; fT\} \in CompFTV$

$\quad \land \neg \{fB; fTrue; fT\} \in CompFTV$

$\quad \land \neg \{fFalse; fTrue; fT\} \in CompFTV$

$\quad \land \neg \{fB; fFalse; fTrue; fT\} \in CompFTV$

$\boldsymbol{CompFTV\_Lub\_lemma} =$

$\vdash \forall s \bullet s \in CompFTV \Leftrightarrow \neg Lub \$\leq_{t4} s = fT$

$\boldsymbol{Lub\_CompFTV\_lemma} =$

$\vdash \forall s \bullet Lub \$\leq_{t4} s = fT \Leftrightarrow \neg s \in CompFTV$

$\boldsymbol{CoCompFTV\_Lub\_lemma} =$

$\vdash \forall s \bullet s \in CoCompFTV \Leftrightarrow \neg Glb \$\leq_{t4} s = fB$

$\boldsymbol{Glb\_CoCompFTV\_lemma} =$

$\vdash \forall s \bullet Glb \$\leq_{t4} s = fB \Leftrightarrow \neg s \in CoCompFTV$

$\boldsymbol{\leq_{t4}\_lin\_lub\_lemma} =$

$\vdash \forall X \bullet LinearOrder (X, \$\leq_{t4}) \Rightarrow fT \leq_{t4} Lub \$\leq_{t4} X = fT \in X$

$\boldsymbol{\leq_{t4}\_lin\_lub\_lemma2} =$

$\vdash \forall X \bullet LinearOrder (X, \$\leq_{t4}) \Rightarrow (fT = Lub \$\leq_{t4} X) = fT \in X$

$\boldsymbol{\leq_{t4}\_lin\_glb\_lemma} =$

$\vdash \forall X \bullet LinearOrder (X, \$\leq_{t4}) \Rightarrow Glb \$\leq_{t4} X \leq_{t4} fB = fB \in X$

$\boldsymbol{\leq_{t4}\_lin\_glb\_lemma2} =$

$\vdash \forall X \bullet LinearOrder (X, \$\leq_{t4}) \Rightarrow (Glb \$\leq_{t4} X = fB) = fB \in X$

4 ORDERS AND PRE-ORDERS

4.1 Domain Restriction

The following operator restricts a reflexive partial order to some subdomain of the type over which it is defined.

SML

```
declare_infix(300, "◁o");
```

HOL Constant

$\$\lhd_o : {'a}\ SET \to ({'a} \to {'a} \to BOOL) \to ({'a} \to {'a} \to BOOL)$

$\forall\ V\ r\bullet\ V\ \lhd_o\ r = \lambda x\ y\bullet\ if\ x \in V \land y \in V\ then\ r\ x\ y\ else\ x = y$

4.2 Conjunction of Orders

HOL Constant

$\textbf{ConjOrder} : ({'a} \to {'a} \to BOOL) \to ({'a} \to {'a} \to BOOL) \to ({'a} \to {'a} \to BOOL)$

$\forall r1\ r2\bullet\ ConjOrder\ r1\ r2 = \lambda x\ y\bullet\ r1\ x\ y \land r2\ x\ y$

4.3 Derived Orderings

I don't know a good name for these, but a common way to impose an order on a collection is by defining a function which maps the collection into some collection for which we have a suitable ordering. Often suitable means well-founded, but in our case it is completeness which is desired.

HOL Constant

$\textbf{DerivedOrder} : ({'b} \to {'a}) \to ({'a} \to {'a} \to BOOL) \to ({'b} \to {'b} \to BOOL)$

$\forall f\ r\bullet\ DerivedOrder\ f\ r = \lambda x\ y\bullet\ r\ (f\ x)\ (f\ y)$

We require sufficient conditions for the result to be complete.

fi_isub_lemma =
$\vdash \forall\ f\ r\ s\ e\bullet\ IsUb\ r\ (FunImage\ f\ s)\ (f\ e) \Rightarrow IsUb\ (DerivedOrder\ f\ r)\ s\ e$

do_isub_lemma =
$\vdash \forall\ f\ r\ s\ x\bullet\ IsUb\ (DerivedOrder\ f\ r)\ s\ x \Rightarrow IsUb\ r\ (FunImage\ f\ s)\ (f\ x)$

fi_islub_lemma =
$\vdash \forall\ f\ r\ s\ e$
$\bullet\ IsLub\ r\ (FunImage\ f\ s)\ (f\ e) \Rightarrow IsLub\ (DerivedOrder\ f\ r)\ s\ e$

do_lubs_exist_thm =
$\vdash \forall\ f\ r\bullet\ LubsExist\ r \land Onto\ f \Rightarrow LubsExist\ (DerivedOrder\ f\ r)$

$\boldsymbol{fi\_islb\_lemma} =$
$\vdash \forall\ f\ r\ s\ e \bullet\ IsLb\ r\ (FunImage\ f\ s)\ (f\ e) \Rightarrow IsLb\ (DerivedOrder\ f\ r)\ s\ e$

$\boldsymbol{do\_islb\_lemma} =$
$\vdash \forall\ f\ r\ s\ x \bullet\ IsLb\ (DerivedOrder\ f\ r)\ s\ x \Rightarrow IsLb\ r\ (FunImage\ f\ s)\ (f\ x)$

$\boldsymbol{fi\_isglb\_lemma} =$
$\vdash \forall\ f\ r\ s\ e \bullet\ IsGlb\ r\ (FunImage\ f\ s)\ (f\ e) \Rightarrow IsGlb\ (DerivedOrder\ f\ r)\ s\ e$

$\boldsymbol{do\_glbs\_exist\_thm} =$
$\vdash \forall\ f\ r \bullet\ GlbsExist\ r \land Onto\ f \Rightarrow GlbsExist\ (DerivedOrder\ f\ r)$

$\boldsymbol{wf\_derived\_order\_thm} =$
$\vdash \forall\ r \bullet\ well\_founded\ r \Rightarrow (\forall\ f \bullet\ well\_founded\ (DerivedOrder\ f\ r))$

4.4 Projections

Projections are a special case of derived orderings in which the onto requirement can be taken for granted.

$\boldsymbol{projections\_onto\_lemma} =$
$\vdash Onto\ Fst \land Onto\ Snd$

$\boldsymbol{lubsexist\_dofst\_thm} =$
$\vdash \forall\ f\ r \bullet\ LubsExist\ r \Rightarrow LubsExist\ (DerivedOrder\ Fst\ r)$

$\boldsymbol{glbsexist\_dofst\_thm} =$
$\vdash \forall\ f\ r \bullet\ GlbsExist\ r \Rightarrow GlbsExist\ (DerivedOrder\ Fst\ r)$

$\boldsymbol{lubsexist\_dosnd\_thm} =$
$\vdash \forall\ f\ r \bullet\ LubsExist\ r \Rightarrow LubsExist\ (DerivedOrder\ Snd\ r)$

$\boldsymbol{glbsexist\_dosnd\_thm} =$
$\vdash \forall\ f\ r \bullet\ GlbsExist\ r \Rightarrow GlbsExist\ (DerivedOrder\ Snd\ r)$

4.5 Functions

Most of our orderings are orderings of functions obtained from orderings of truth values by the following operation.

HOL Constant

$\boldsymbol{Pw} : ('a \to 'a \to BOOL) \to (('b \to 'a) \to ('b \to 'a) \to BOOL)$

$\forall\ r \bullet\ Pw\ r = \lambda\ lo\ ro \bullet\ \forall x \bullet\ r\ (lo\ x)\ (ro\ x)$

$\boldsymbol{pw\_isub\_lemma} =$

$\vdash \forall\ r\ G\ f \bullet (\forall\ v \bullet\ IsUb\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v)) \Rightarrow IsUb\ (Pw\ r)\ G\ f$

$\boldsymbol{pw\_isub\_lemma2} =$

$\vdash \forall\ r\ G\ f \bullet\ IsUb\ (Pw\ r)\ G\ f \Leftrightarrow (\forall\ v \bullet\ IsUb\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v))$

$\boldsymbol{pw\_islb\_lemma} =$

$\vdash \forall\ r\ G\ f \bullet (\forall\ v \bullet\ IsLb\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v)) \Rightarrow IsLb\ (Pw\ r)\ G\ f$

$\boldsymbol{pw\_islb\_lemma2} =$

$\vdash \forall\ r\ G\ f \bullet\ IsLb\ (Pw\ r)\ G\ f \Leftrightarrow (\forall\ v \bullet\ IsLb\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v))$

$\boldsymbol{pw\_islub\_lemma} =$

$\vdash \forall\ r\ G\ f$

$\bullet (\forall\ v \bullet\ IsLub\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v)) \Rightarrow IsLub\ (Pw\ r)\ G\ f$

$\boldsymbol{pw\_islub\_lemma2} =$

$\vdash \forall\ r\ G\ f$

$\bullet\ IsLub\ (Pw\ r)\ G\ f \Leftrightarrow (\forall\ v \bullet\ IsLub\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v))$

$\boldsymbol{pw\_isglb\_lemma} =$

$\vdash \forall\ r\ G\ f$

$\bullet (\forall\ v \bullet\ IsGlb\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v)) \Rightarrow IsGlb\ (Pw\ r)\ G\ f$

$\boldsymbol{pw\_isglb\_lemma2} =$

$\vdash \forall\ r\ G\ f$

$\bullet\ IsGlb\ (Pw\ r)\ G\ f \Leftrightarrow (\forall\ v \bullet\ IsGlb\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v))$

$\boldsymbol{pw\_rpo\_lemma} =$

$\vdash \forall\ r \bullet\ Rpo\ (Universe,\ r) \Rightarrow Rpo\ (Universe,\ Pw\ r)$

$\boldsymbol{pw\_cc\_lemma} =$

$\vdash \forall\ r \bullet\ CcRpo\ (Universe,\ r) \Rightarrow ChainComplete\ (Universe,\ Pw\ r)$

$\boldsymbol{pw\_ccrpou\_thm} =$

$\vdash \forall\ r \bullet\ CcRpoU\ r \Rightarrow CcRpoU\ (Pw\ r)$

$\boldsymbol{pw\_lubs\_exist\_thm} =$

$\vdash \forall\ r \bullet\ LubsExist\ r \Rightarrow LubsExist\ (Pw\ r)$

$\boldsymbol{pw\_glbs\_exist\_thm} =$

$\vdash \forall\ r \bullet\ GlbsExist\ r \Rightarrow GlbsExist\ (Pw\ r)$

$\boldsymbol{pw\_crpou\_thm} =$

$\vdash \forall\ r \bullet\ CRpoU\ r \Rightarrow CRpoU\ (Pw\ r)$

$pw\_\leq_{t4}\_lubsexist\_thm = \vdash LubsExist\ (Pw\ \$\leq_{t4})$

$pw\_\leq_{t4}\_glbsexist\_thm = \vdash GlbsExist\ (Pw\ \$\leq_{t4})$

$pw\_\leq_{t4}\_crpou\_thm = \vdash CRpoU\ (Pw\ \$\leq_{t4})$

crpou_lub_pw_lemma =
$\vdash \forall r$
$\bullet CRpoU\ r \Rightarrow (\forall G \bullet Lub\ (Pw\ r)\ G = (\lambda x \bullet Lub\ r\ \{w | \exists g \bullet g \in G \wedge w = g\ x\}))$

crpou_lub_pw_pw_lemma =
$\vdash \forall r$
$\bullet CRpoU\ r$
$\Rightarrow (\forall G$
$\bullet Lub\ (Pw\ (Pw\ r))\ G = (\lambda x\ y \bullet Lub\ r\ \{w | \exists g \bullet g \in G \wedge w = g\ x\ y\}))$

Sometimes we are only interested in the behaviour of a function over some subset of its domain type. The following version of *Pw* is parameterised by a set which gives the subdomain over which the values of the function are significant to the resulting ordering.

This is not a partial order since it is not antisymmetric. It is a pre-order.

HOL Constant

$\boldsymbol{PwS} : 'b\ SET \to ('a \to 'a \to BOOL) \to (('b \to 'a) \to ('b \to 'a) \to BOOL)$

$\forall V\ r \bullet PwS\ V\ r = \lambda lo\ ro \bullet \forall x \bullet x \in V \Rightarrow r\ (lo\ x)\ (ro\ x)$

pws_isub_lemma =
$\vdash \forall V\ r\ G\ f \bullet (\forall v \bullet v \in V \Rightarrow IsUb\ r\ \{w \mid \exists g \bullet g \in G \wedge w = g\ v\}\ (f\ v))$
$\Rightarrow IsUb\ (PwS\ V\ r)\ G\ f$

pws_isub_lemma3 =
$\vdash \forall V\ r\ G\ f$
$\bullet IsUb\ (PwS\ V\ r)\ G\ f$
$\Leftrightarrow (\forall v \bullet v \in V \Rightarrow IsUb\ r\ \{w | \exists g \bullet g \in G \wedge w = g\ v\}\ (f\ v))$

pws_islb_lemma =
$\vdash \forall V\ r\ G\ f \bullet (\forall v \bullet v \in V \Rightarrow IsLb\ r\ \{w | \exists g \bullet g \in G \wedge w = g\ v\}\ (f\ v))$
$\Rightarrow IsLb\ (PwS\ V\ r)\ G\ f$

pws_islb_lemma2 =
$\vdash \forall V\ r\ G\ f$
$\bullet IsLb\ (PwS\ V\ r)\ G\ f$
$\Rightarrow (\forall v \bullet v \in V \Rightarrow IsLb\ r\ \{w | \exists g \bullet g \in G \wedge w = g\ v\}\ (f\ v))$

pws_islb_lemma3 =

$\vdash \forall\ V\ r\ G\ f$
$\bullet\ IsLb\ (PwS\ V\ r)\ G\ f$
$\Leftrightarrow (\forall\ v \bullet\ v \in V \Rightarrow IsLb\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v))$

pws_islub_lemma =
$\vdash \forall\ V\ r\ G\ f$
$\bullet\ (\forall\ v \bullet\ v \in V \Rightarrow IsLub\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v))$
$\Rightarrow IsLub\ (PwS\ V\ r)\ G\ f$

pws_islub_lemma2 =
$\vdash \forall\ V\ r\ G\ f$
$\bullet\ IsLub\ (PwS\ V\ r)\ G\ f$
$\Rightarrow (\forall\ v \bullet\ v \in V \Rightarrow IsLub\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v))$

pws_islub_lemma3 =
$\vdash \forall\ V\ r\ G\ f$
$\bullet\ IsLub\ (PwS\ V\ r)\ G\ f$
$\Leftrightarrow (\forall\ v \bullet\ v \in V \Rightarrow IsLub\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v))$

pws_isglb_lemma =
$\vdash \forall\ V\ r\ G\ f$
$\bullet\ (\forall\ v \bullet\ v \in V \Rightarrow IsGlb\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v))$
$\Rightarrow IsGlb\ (PwS\ V\ r)\ G\ f$

pws_isglb_lemma2 =
$\vdash \forall\ V\ r\ G\ f$
$\bullet\ IsGlb\ (PwS\ V\ r)\ G\ f$
$\Rightarrow (\forall\ v \bullet\ v \in V \Rightarrow IsGlb\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v))$

pws_isglb_lemma3 =
$\vdash \forall\ V\ r\ G\ f$
$\bullet\ (\forall\ v \bullet\ v \in V \Rightarrow IsGlb\ r\ \{w | \exists\ g \bullet\ g \in G \land w = g\ v\}\ (f\ v))$
$\Leftrightarrow IsGlb\ (PwS\ V\ r)\ G\ f$

pws_lubs_exist_thm =
$\vdash \forall\ V\ r \bullet\ LubsExist\ r \Rightarrow LubsExist\ (PwS\ V\ r)$

pws_glbs_exist_thm =
$\vdash \forall\ V\ r \bullet\ GlbsExist\ r \Rightarrow GlbsExist\ (PwS\ V\ r)$

4.6 Products

We also need the following ordering over products.

HOL Constant

$$\boldsymbol{PrO} : ('a \to 'a \to BOOL) \to ('b \to 'b \to BOOL) \to (('a \times 'b) \to ('a \times 'b) \to BOOL)$$

$$\forall\ ra\ rb\bullet\ PrO\ ra\ rb = \lambda l\ r\bullet\ ra\ (Fst\ l)\ (Fst\ r) \land rb\ (Snd\ l)\ (Snd\ r)$$

$\boldsymbol{pro\_isub\_lemma} =$

$$\vdash \forall\ ra\ rb\ s\ lub\ rub \bullet IsUb\ ra\ \{w|\exists\ g\bullet\ g \in s \land w = Fst\ g\}\ lub \land IsUb\ rb\ \{w|\exists\ g\bullet\ g \in s \land w = Snd\ g\}\ rub \Leftrightarrow IsUb\ (PrO\ ra\ rb)\ s\ (lub, rub)$$

$\boldsymbol{pro\_islb\_lemma} =$

$$\vdash \forall\ ra\ rb\ s\ llb\ rlb \bullet IsLb\ ra\ \{w|\exists\ g\bullet\ g \in s \land w = Fst\ g\}\ llb \land IsLb\ rb\ \{w|\exists\ g\bullet\ g \in s \land w = Snd\ g\}\ rlb \Leftrightarrow IsLb\ (PrO\ ra\ rb)\ s\ (llb, rlb)$$

$\boldsymbol{pro\_islub\_lemma} =$

$$\vdash \forall\ ra\ rb\ s\ llub\ rlub \bullet IsLub\ ra\ \{w|\exists\ g\bullet\ g \in s \land w = Fst\ g\}\ llub \land IsLub\ rb\ \{w|\exists\ g\bullet\ g \in s \land w = Snd\ g\}\ rlub \Rightarrow IsLub\ (PrO\ ra\ rb)\ s\ (llub, rlub)$$

$\boldsymbol{pro\_isglb\_lemma} =$

$$\vdash \forall\ ra\ rb\ s\ lglb\ rglb \bullet IsGlb\ ra\ \{w|\exists\ g\bullet\ g \in s \land w = Fst\ g\}\ lglb \land IsGlb\ rb\ \{w|\exists\ g\bullet\ g \in s \land w = Snd\ g\}\ rglb \Rightarrow IsGlb\ (PrO\ ra\ rb)\ s\ (lglb, rglb)$$

$\boldsymbol{pro\_refl\_lemma} =$

$$\vdash \forall\ ra\ rb \bullet Refl\ (Universe, ra) \land Refl\ (Universe, rb) \Rightarrow Refl\ (Universe, PrO\ ra\ rb)$$

$\boldsymbol{pro\_partialorder\_lemma} =$

$$\vdash \forall\ ra\ rb \bullet PartialOrder\ (Universe, ra) \land PartialOrder\ (Universe, rb) \Rightarrow PartialOrder\ (Universe, PrO\ ra\ rb)$$

$\boldsymbol{pro\_rpo\_lemma} =$

$$\vdash \forall\ ra\ rb \bullet Rpo\ (Universe, ra) \land Rpo\ (Universe, rb) \Rightarrow Rpo\ (Universe, PrO\ ra\ rb)$$

$\boldsymbol{pro\_lubs\_exist\_thm} =$

$$\vdash \forall\ ra\ rb\bullet\ LubsExist\ ra \land LubsExist\ rb \Rightarrow LubsExist\ (PrO\ ra\ rb)$$

$pro\_glbs\_exist\_thm$ =
$\vdash \forall\ ra\ rb\bullet\ GlbsExist\ ra \land GlbsExist\ rb \Rightarrow GlbsExist\ (PrO\ ra\ rb)$

pro_crpou_thm =
$\vdash \forall\ ra\ rb\bullet\ CRpoU\ ra \land CRpoU\ rb \Rightarrow CRpoU\ (PrO\ ra\ rb)$

4.7 OPT

HOL Constant

$OptO : ('a \to 'a \to BOOL) \to ('a\ OPT \to 'a\ OPT \to BOOL)$

$\forall rl\bullet\ OptO\ rl = \lambda l\ r\bullet\ l = Undefined$
$\quad \lor IsDefined\ l \land IsDefined\ r \land rl\ (ValueOf\ l)\ (ValueOf\ r)$

4.8 Discrete Partial Orders

The things I am calling discrete partial orders come with the ordering of a discrete lattice.

When these are used to construct indexed sets whose elements have some ordering then there will be another ordering which is derived from the ordering on the elements. This is defined for use in defining orderings over indexed sets.

HOL Constant

$DpoEO : ('a \to 'a \to BOOL) \to ('a\ DPO \to 'a\ DPO \to BOOL)$

$\forall rl\bullet\ DpoEO\ rl = \lambda l\ r\bullet\ dpoUdef\ l \lor dpoOdef\ r$
$\quad \lor \exists le\ re\bullet\ rl\ le\ re \land l = dpoE\ le \land r = dpoE\ re$

To get the discrete ordering apply this function to the equality relation.

is_isub_lemma =
$\vdash \forall\ r\ G\ d$
$\bullet\ IsUb\ r\ G\ d \Rightarrow IsUb\ (DpoEO\ r)\ \{w|\exists\ v\bullet\ v \in G \land w = dpoE\ v\}\ (dpoE\ d)$

is_islb_lemma =
$\vdash \forall\ r\ G\ d$
$\bullet\ IsLb\ r\ G\ d \Rightarrow IsLb\ (DpoEO\ r)\ \{w|\exists\ v\bullet\ v \in G \land w = dpoE\ v\}\ (dpoE\ d)$

is_isub_lemma2 =
$\vdash \forall\ r\ G\ d$
$\bullet\ IsUb\ (DpoEO\ r)\ G\ d$
$\quad = (d = dpoT$
$\quad \lor G \subseteq \{dpoB\} \land d = dpoB$
$\quad \lor (\exists\ e\bullet\ d = dpoE\ e \land IsUb\ r\ \{w|dpoE\ w \in G\}\ e) \land \neg\ dpoT \in G)$

is_islb_lemma2 =
$\vdash \forall\ r\ G\ d$

$\bullet\ IsLb\ (DpoEO\ r)\ G\ d$
$= (d = dpoB$
$\lor\ G \subseteq \{dpoT\} \land d = dpoT$
$\lor\ (\exists\ e\bullet\ d = dpoE\ e \land IsLb\ r\ \{w|dpoE\ w \in G\}\ e) \land \neg\ dpoB \in G)$

is_islub_lemma =

$\vdash \forall\ r\ G\ d$
$\bullet\ IsLub\ (DpoEO\ r)\ G\ d$
$= ((dpoT \in G \lor (\exists\ w\bullet\ dpoE\ w \in G) \land \neg\ (\exists\ e\bullet\ IsUb\ r\ \{w|dpoE\ w \in G\}\ e))$
$\land\ d = dpoT$
$\lor\ G \subseteq \{dpoB\} \land d = dpoB$
$\lor\ \neg\ G \subseteq \{dpoB\}$
$\land\ (\exists\ e\bullet\ d = dpoE\ e \land IsLub\ r\ \{w|dpoE\ w \in G\}\ e)$
$\land\ \neg\ dpoT \in G)$

is_isglb_lemma =

$\vdash \forall\ r\ G\ d$
$\bullet\ IsGlb\ (DpoEO\ r)\ G\ d$
$= ((dpoB \in G \lor (\exists\ w\bullet\ dpoE\ w \in G) \land \neg\ (\exists\ e\bullet\ IsLb\ r\ \{w|dpoE\ w \in G\}\ e))$
$\land\ d = dpoB$
$\lor\ G \subseteq \{dpoT\} \land d = dpoT$
$\lor\ \neg\ G \subseteq \{dpoT\}$
$\land\ (\exists\ e\bullet\ d = dpoE\ e \land IsGlb\ r\ \{w|dpoE\ w \in G\}\ e)$
$\land\ \neg\ dpoB \in G)$

$\le_{t4}$_trich_lub_ft_lemma =

$\vdash \forall\ X\ z\bullet\ Trich\ (X, \$\le_{t4}) \land fT \le_{t4}\ Lub\ \$\le_{t4}\ X \Rightarrow fT \in X$

4.9 Indexed Sets

4.9.1 IX

The following function lifts an ordering on the elements of the codomain to an ordering on the indexed sets.

HOL Constant

$\boldsymbol{IxO} : ('b \to 'b \to BOOL) \to (('a, 'b)\ IX \to ('a, 'b)\ IX \to BOOL)$

$\forall r\bullet\ IxO\ r = Pw\ (OptO\ r)$

It may be more convenient in some cases to use the following ordering in which the comparison is restricted to some subdomain.

(I should have used domain restriction here.)

HOL Constant

$$\mathbf{IxO2} : ('a, 'b)\ IX\ SET \times ('b \to 'b \to BOOL) \to (('a, 'b)\ IX \to ('a, 'b)\ IX \to BOOL)$$

$$\begin{array}{l} \forall d\ r \bullet\ IxO2\ (d,\ r) = \lambda x\ y \bullet \\ \quad\quad if\ x \in d \land y \in d \\ \quad\quad then\ IxDom\ x = IxDom\ y \land IxO\ r\ x\ y \\ \quad\quad else\ x = y \end{array}$$

4.10 A Pre-order on Sets

The following pre-order on sets based on a pre-order of the elements is used later with sets of truth values.

HOL Constant

$$\mathbf{SetO} : ('a \to 'a \to BOOL) \to ('a\ SET) \to ('a\ SET) \to BOOL$$

$$\begin{array}{l} \forall\ r \bullet\ SetO\ r = \lambda m\ n \bullet \\ \quad\quad (\forall x \bullet\ x \in m \Rightarrow \exists y \bullet\ y \in n \land r\ x\ y) \\ \quad \land \quad (\forall y \bullet\ y \in n \Rightarrow \exists x \bullet\ x \in m \land r\ x\ y) \end{array}$$

That turns out to be stronger than we need, this one simplifies matters.

HOL Constant

$$\mathbf{SetO2} : ('a \to 'a \to BOOL) \to ('a\ SET) \to ('a\ SET) \to BOOL$$

$$\forall\ r \bullet\ SetO2\ r = \lambda m\ n \bullet\ \forall x \bullet\ x \in m \Rightarrow \exists y \bullet\ r\ x\ y \land y \in n$$

trans_seto_lemma =

$$\vdash \forall\ r \bullet\ Trans\ (Universe,\ r) \Rightarrow Trans\ (Universe,\ SetO\ r)$$

trans_seto2_lemma =

$$\vdash \forall\ r \bullet\ Trans\ (Universe,\ r) \Rightarrow Trans\ (Universe,\ SetO2\ r)$$

4.11 A Pre-order for Sets of Functions

To get a pre-order over sets of functions from an pre-order of the codomain of the functions, we could apply *Pw* and then *SetO*, however the following construction proves more useful.

[Though I don't appear to have used it!]

HOL Constant

$$\mathbf{FunSetO} : ('a \to 'a \to BOOL) \to ('b \to 'a)\ SET \to ('b \to 'a)\ SET \to BOOL$$

$$\begin{array}{l} \forall\ r{:}('a \to 'a \to BOOL) \bullet\ FunSetO\ r = \\ \quad \lambda s\ t \bullet\ Pw\ (SetO\ r)\ (\lambda x \bullet\ \{v \mid \exists y \bullet\ y \in s \land v = y\ x\})\ (\lambda x \bullet\ \{v \mid \exists y \bullet\ y \in t \land v = y\ x\}) \end{array}$$

5 MISCELLANEOUS THEOREMS

5.1 Monotonicity of Lub

$\textbf{\textit{lub\_increasing\_lemma}} =$
$\vdash \forall\ r \bullet\ RpoU\ r \land LubsExist\ r \Rightarrow Increasing\ (SetO\ r)\ r\ (Lub\ r)$

$\textbf{\textit{lub\_increasing2\_lemma}} =$
$\vdash \forall\ r \bullet\ RpoU\ r \land LubsExist\ r \Rightarrow Increasing\ (SetO2\ r)\ r\ (Lub\ r)$

$\textbf{\textit{lub\_increasing\_lemma2}} =$
$\vdash \forall\ r \bullet\ CRpoU\ r \Rightarrow Increasing\ (SetO\ r)\ r\ (Lub\ r)$

$\textbf{\textit{lub\_increasing2\_lemma2}} =$
$\vdash \forall\ r \bullet\ CRpoU\ r \Rightarrow Increasing\ (SetO2\ r)\ r\ (Lub\ r)$

$\textbf{\textit{lub\_increasing\_lemma3}} =$
$\vdash \forall\ r \bullet\ CRpoU\ r \Rightarrow (\forall\ x\ y \bullet\ SetO\ r\ x\ y \Rightarrow r\ (Lub\ r\ x)\ (Lub\ r\ y))$

$\textbf{\textit{lub\_increasing2\_lemma3}} =$
$\vdash \forall\ r \bullet\ CRpoU\ r \Rightarrow (\forall\ x\ y \bullet\ SetO2\ r\ x\ y \Rightarrow r\ (Lub\ r\ x)\ (Lub\ r\ y))$

5.2 Product of Functions

We now define the product of two functions:

HOL Constant

$\textbf{\textit{FunProd}} : ('a \to 'b) \to ('a \to 'c) \to ('a \to 'b \times 'c)$

$\forall\ f\ g \bullet\ FunProd\ f\ g = \lambda x \bullet\ (f\ x,\ g\ x)$

And prove that the product of two increasing functions is increasing.

$\textbf{\textit{funprod\_increasing\_thm}} =$
$\vdash \forall\ f\ g\ ra\ rb\ rc$
$\bullet\ Increasing\ ra\ rb\ f \land Increasing\ ra\ rc\ g$
$\Rightarrow Increasing\ ra\ (PrO\ rb\ rc)\ (FunProd\ f\ g)$

HOL Constant

$\textbf{\textit{FunLeft}} : ('a \times 'c \to 'b) \to ('a \times 'c \to 'b \times 'c)$

$\forall\ f \bullet\ FunLeft\ f = \lambda x \bullet\ (f\ x,\ Snd\ x)$

HOL Constant

$\textbf{\textit{FunRight}} : ('c \times 'a \to 'b) \to ('c \times 'a \to 'c \times 'b)$

$\forall\ f \bullet\ FunRight\ f = \lambda x \bullet\ (Fst\ x,\ f\ x)$

$\textit{\textbf{funleft\_increasing\_thm}} =$
$\vdash \forall\ f\ ra\ rb\ rc$
$\bullet\ Increasing\ (PrO\ ra\ rc)\ rb\ f$
$\Rightarrow Increasing\ (PrO\ ra\ rc)\ (PrO\ rb\ rc)\ (FunLeft\ f)$

$\textit{\textbf{funright\_increasing\_thm}} =$
$\vdash \forall\ f\ ra\ rb\ rc$
$\bullet\ Increasing\ (PrO\ rc\ ra)\ rb\ f$
$\Rightarrow Increasing\ (PrO\ rc\ ra)\ (PrO\ rc\ rb)\ (FunRight\ f)$

5.3 FunImage Preserves Linearity

The function must be a morphism and the target set must be a partial order (these are sufficient conditions).

$\textit{\textbf{trich\_funimage\_lemma}} =$
$\vdash \forall\ r1\ r2\ f\ X \bullet\ Increasing\ r1\ r2\ f \Rightarrow Trich\ (X,\ r1) \Rightarrow Trich\ (FunImage\ f\ X,\ r2)$

$\textit{\textbf{linear\_funimage\_thm}} =$
$\vdash \forall\ r1\ r2\ f\ X \bullet\ Increasing\ r1\ r2\ f$
$\wedge\ LinearOrder\ (X,\ r1)$
$\wedge\ PartialOrder\ (FunImage\ f\ X,\ r2)$
$\Rightarrow LinearOrder\ (FunImage\ f\ X,\ r2)$

6 GENERALISED RELATIONS

SML

```
declare_type_abbrev ("BR", ["'a", "'b"], ⌜:'a → 'a → 'b⌝);
declare_infix (300, "≤∈");
```

6.1 Partial Relations

One way to represent a partial relation is to use more than one truth value. Alternatively you can have two relations, one for the true values and one for the false ones.

In the latter case you might have an inconsistency, a pair might appear in both. This ‘defect’ is also present if you chose the former representation using four truth values as in a discrete partial ordering of the type BOOL.

We might as well have a type abbreviation for the partial (equivalence) relations.

SML

```
declare_infix (300, "=_p");
declare_infix (300, "=_q");
declare_infix (300, "=_r");
```

It will be convenient perhaps to be able to switch between having a single four valued relation and having two boolean relations.

HOL Constant

$$\textbf{Pr2BrT} : ('a,\ FTV)\ BR \to ('a,\ BOOL)\ BR$$

$$\forall \$=_p \bullet\ Pr2BrT\ \$=_p\ =\ \lambda x\ y \bullet\ \ fTrue\ \leq_{t4}\ (x\ =_p\ y)$$

HOL Constant

$$\textbf{Pr2BrF} : ('a,\ FTV)\ BR \to ('a,\ BOOL)\ BR$$

$$\forall \$=_p \bullet\ Pr2BrF\ \$=_p\ =\ \lambda x\ y \bullet\ \ fFalse\ \leq_{t4}\ (x\ =_p\ y)$$

HOL Constant

$$\textbf{BrTF2Pr} : ('a,\ BOOL)\ BR \to ('a,\ BOOL)\ BR \to ('a,\ FTV)\ BR$$

$$\forall \$=_p\ \$=_q \bullet\ BrTF2Pr\ \$=_p\ \$=_q\ =\ \lambda x\ y \bullet\ Lub\ \$\leq_{t4}\ \{ftv\ |\ x\ =_p\ y\ \wedge\ ftv\ =\ fTrue\ \vee\ x\ =_p\ y\ \wedge\ ftv\ =\ fFal$$

6.2 Proof Context

SML

```
add_pc_thms "'misc1" [];
commit_pc "'misc1";

force_new_pc "misc1";
merge_pcs ["rbjmisc", "'misc1"] "misc1";
commit_pc "misc1";
force_new_pc "misc11";
merge_pcs ["rbjmisc1", "'misc1"] "misc11";
commit_pc "misc11";
```

7 MISC2

SML

```
open_theory "misc1";
force_new_theory "misc2";
new_parent "GS";
force_new_pc "'misc2";
merge_pcs ["'savedthm_cs_∃_proof"] "'misc2";
set_merge_pcs ["misc1", "'GS1", "'misc2"];
```

8 SET THEORY

8.1 Mapping Functions over Sets

The following function makes recursive definition of functions over sets of type GS just a little more compact.

HOL Constant

$$\boldsymbol{FunImage_g} : (GS \to {'a}) \to GS \to ({'a}\ SET)$$

$$\forall f\ s \bullet\ FunImage_g\ f\ s = \{x \mid \exists y \bullet\ y \in_g s \wedge x = f\ y\}$$

$$\boldsymbol{funimage_g\_fc\_lemma} =$$
$$\vdash \forall\ f\ s\ x \bullet\ x \in_g s \Rightarrow f\ x \in FunImage_g\ f\ s$$

9 INDEXED SETS

There is a version of indexed sets in [1].

In this version the functions yield DPOs. This gives a complete partial order over the indexed sets which was required for some versions of infinitary first order logic. This is more complicated of course and not to be used unless essential.

In this implementation of indexed sets we use discrete partial orders in the codomain, so that the resulting partial orders are complete.

SML

```
declare_type_abbrev("IS", ["'a"], ⌜:GS → 'a DPO⌝);
```

HOL Constant

$$\boldsymbol{IsVal} : {'a}\ IS \to GS \to {'a}$$

$$\forall is\ g \bullet\ IsVal\ is\ g = dpoV\ (is\ g)$$

HOL Constant

$$\boldsymbol{IsRan} : {'a}\ IS \to {'a}\ SET$$

$$\forall is \bullet\ IsRan\ is = \{v \mid \exists \alpha \bullet\ dpoE\ v = is\ \alpha\}$$

HOL Constant

$$\boldsymbol{IsDom} : {'a}\ IS \to GS\ SET$$

$$\forall is \bullet\ IsDom\ is = \{i \mid \neg\ (is\ i) = dpoB\}$$

HOL Constant

$$\boldsymbol{IsSDom} : {'a}\ IS \to GS\ SET$$

$$\forall is \bullet\ IsSDom\ is = \{i \mid \neg\ ((is\ i) = dpoB \vee (is\ i) = dpoT)\}$$

$$\boldsymbol{is\_domran\_lemma} =$$
$$\vdash \forall\ x\ y \bullet\ x \in IsSDom\ y \Rightarrow IsVal\ y\ x \in IsRan\ y$$

HOL Constant

$\boldsymbol{IsOd} : {'a}\ IS \to GS\ SET$

$\forall is \bullet\ IsOd\ is = \{i \mid is\ i = dpoT\}$

HOL Constant

$\boldsymbol{IsUd} : {'a}\ IS \to GS\ SET$

$\forall is \bullet\ IsUd\ is = \{i \mid is\ i = dpoB\}$

HOL Constant

$\boldsymbol{IsTDom} : {'a}\ IS \to (GS\ SET \times GS\ SET \times GS\ SET)$

$\forall is \bullet\ IsTDom\ is = (IsSDom\ is,\ IsUd\ is,\ IsOd\ is)$

HOL Constant

$\boldsymbol{IsOverRide} : {'a}\ IS \to {'a}\ IS \to {'a}\ IS$

$\forall is1\ is2 \bullet\ IsOverRide\ is1\ is2 =$
$\quad \lambda i \bullet\ if\ \neg\ dpoUdef\ (is2\ i)\ then\ is2\ i\ else\ is1\ i$

$\boldsymbol{istdom\_eq\_fc\_lemma} =$
$\quad \vdash \forall\ x\ y$
$\quad\quad \bullet\ IsTDom\ x = IsTDom\ y$
$\quad\quad\quad \Rightarrow IsUd\ x = IsUd\ y$
$\quad\quad\quad\quad \wedge\ IsOd\ x = IsOd\ y$
$\quad\quad\quad\quad \wedge\ IsSDom\ x = IsSDom\ y$
$\quad\quad\quad\quad \wedge\ IsDom\ x = IsDom\ y$

$\boldsymbol{isoverride\_isdom\_lemma} =$
$\quad \vdash \forall\ x\ y \bullet\ IsDom\ (IsOverRide\ x\ y) = IsDom\ x \cup IsDom\ y$

$\boldsymbol{isoverride\_isud\_lemma} =$
$\quad \vdash \forall\ x\ y \bullet\ IsUd\ (IsOverRide\ x\ y) = IsUd\ x \setminus IsDom\ y$

$\boldsymbol{isoverride\_isod\_lemma} =$
$\quad \vdash \forall\ x\ y \bullet\ IsOd\ (IsOverRide\ x\ y) = IsOd\ y \cup IsOd\ x \setminus IsDom\ y$

$isoverride\_issdom\_lemma =$
$\quad \vdash \forall\ x\ y \bullet\ IsSDom\ (IsOverRide\ x\ y) = IsSDom\ y \cup IsSDom\ x \setminus IsDom\ y$

10 ORDERS AND PRE-ORDERS

10.1 Indexed Sets

10.1.1 IS

Indexed sets are functions whose codomain is a discrete partial order. From any ordering of the codomain an ordering of the indexed sets may be obtained using *Pw*. This can be done with the discrete order, but we also need to do this with other orders.

The following function lifts an ordering on the elements of the codomain to an ordering on the indexed sets.

HOL Constant

$\boldsymbol{IsEO} : ('a \to 'a \to BOOL) \to ('a\ IS \to 'a\ IS \to BOOL)$

$\forall r \bullet\ IsEO\ r = Pw\ (DpoEO\ r)$

$\boldsymbol{is\_lubs\_exist\_thm} =$
$\quad \vdash \forall\ r \bullet\ LubsExist\ r \Rightarrow LubsExist\ (IsEO\ r)$

$\boldsymbol{is\_glbs\_exist\_thm} =$
$\quad \vdash \forall\ r \bullet\ GlbsExist\ r \Rightarrow GlbsExist\ (IsEO\ r)$

10.2 Partial Relations

For this section a partial relation is taken to be a four valued relation. In my applications these are membership relations so the ordering is suggestive of those applications.

HOL Constant

$\boldsymbol{\$\leq_\in} : (GS,\ FTV)BR \to (GS,\ FTV)BR \to BOOL$

$\$\leq_\in\ = Pw\ (Pw\ \$\leq_{t4})$

$\boldsymbol{crpou\_\leq_\in\_thm} = \vdash\ CRpoU\ \$\leq_\in$

$\boldsymbol{ccrpou\_\leq_\in\_thm} = \vdash\ CcRpoU\ \$\leq_\in$

$\boldsymbol{\leq_\in\_clauses} = \vdash\ GlbsExist\ \$\leq_\in \wedge LubsExist\ \$\leq_\in \wedge (\forall\ x \bullet\ x \leq_\in x)$

$\boldsymbol{\leq_\in\_fc\_clauses} =$
$\quad \vdash (\forall\ x\ y\ z \bullet\ x \leq_\in y \wedge y \leq_\in z \Rightarrow x \leq_\in z)$
$\quad\quad \wedge (\forall\ x\ y \bullet\ x \leq_\in y \wedge y \leq_\in x \Rightarrow x = y)$

Because the ordering here is derived from the ordering on the four truth values there are some simplifications to reasoning about limits which are worth turning into theorems.

$\boldsymbol{\leq_\in\_lub\_thm} =$
$\quad \vdash \forall\ G \bullet\ Lub\ \$\leq_\in\ G = (\lambda\ x\ y \bullet\ Lub\ \$\leq_{t4}\ \{w | \exists\ g \bullet\ g \in G \wedge w = g\ x\ y\})$

11 MISCELLANEOUS THEOREMS

11.1 Partial Relations

$\leq_{\in}$_***increasing_pointwise_thm*** =
⊢ *Increasing* $\$\leq_{\in}$ $\$\leq_{t4}$ $(\lambda\ f \bullet\ f\ x'\ y)$

11.2 Proof Contexts

SML

```
add_pc_thms "'misc2" [];
commit_pc "'misc2";

force_new_pc "misc2";
merge_pcs ["misc1", "'GS1", "'misc2"] "misc2";
commit_pc "misc2";
force_new_pc "misc21";
merge_pcs ["misc11", "'GS1", "'misc2"] "misc21";
commit_pc "misc21";
```

12 The Theory misc1

12.1 Parents

fixp *rbjmisc*

12.2 Children

misc2

12.3 Constants

repDPO	$'a\ DPO \to 'a + BOOL$
absDPO	$'a + BOOL \to 'a\ DPO$
dpoB	$'a\ DPO$
dpoT	$'a\ DPO$
dpoE	$'a \to 'a\ DPO$
dpoV	$'a\ DPO \to 'a$
dpoUdef	$'a\ DPO \to BOOL$
dpoOdef	$'a\ DPO \to BOOL$
Dpo	$('a\ DPO, BOOL)\ BR$
pTrue	TTV
pFalse	TTV
pU	TTV
$\$\le_{t3}$	$(TTV, BOOL)\ BR$
fTrue	FTV
fFalse	FTV
fB	FTV
fT	FTV
$\$\le_{t4}$	$(FTV, BOOL)\ BR$
CompFTV	$FTV\ \mathbb{P}\ \mathbb{P}$
CoCompFTV	$FTV\ \mathbb{P}\ \mathbb{P}$
$\$\lhd_o$	$'a\ \mathbb{P} \to ('a, BOOL)\ BR \to ('a, BOOL)\ BR$
ConjOrder	$(('a, BOOL)\ BR, ('a, BOOL)\ BR)\ BR$
DerivedOrder	$('b \to 'a) \to ('a, BOOL)\ BR \to ('b, BOOL)\ BR$
Pw	$('a, BOOL)\ BR \to ('b \to 'a, BOOL)\ BR$
PwS	$'b\ \mathbb{P} \to ('a, BOOL)\ BR \to ('b \to 'a, BOOL)\ BR$
PrO	$('a, BOOL)\ BR \to ('b, BOOL)\ BR \to ('a \times 'b, BOOL)\ BR$
OptO	$('a, BOOL)\ BR \to ('a\ OPT, BOOL)\ BR$
DpoEO	$('a, BOOL)\ BR \to ('a\ DPO, BOOL)\ BR$
IxO	$('b, BOOL)\ BR \to (('a, 'b)\ IX, BOOL)\ BR$
IxO2	$('a, 'b)\ IX\ \mathbb{P} \times ('b, BOOL)\ BR \to (('a, 'b)\ IX, BOOL)\ BR$
SetO	$('a, BOOL)\ BR \to ('a\ \mathbb{P}, BOOL)\ BR$
SetO2	$('a, BOOL)\ BR \to ('a\ \mathbb{P}, BOOL)\ BR$
FunSetO	$('a, BOOL)\ BR \to (('b \to 'a)\ \mathbb{P}, BOOL)\ BR$
FunProd	$('a \to 'b) \to ('a \to 'c) \to 'a \to 'b \times 'c$
FunLeft	$('a \times 'c \to 'b) \to 'a \times 'c \to 'b \times 'c$
FunRight	$('c \times 'a \to 'b) \to 'c \times 'a \to 'c \times 'b$
Pr2BrT	$('a, FTV)\ BR \to ('a, BOOL)\ BR$
Pr2BrF	$('a, FTV)\ BR \to ('a, BOOL)\ BR$
BrTF2Pr	$(('a, BOOL)\ BR, ('a, FTV)\ BR)\ BR$

12.4 Types

$'1$ ***DPO***

12.5 Type Abbreviations

TTV	*TTV*
$'a$ ***REL***	$('a,\ BOOL)\ BR$
FTV	*FTV*
$('a, 'b)$ ***BR***	$('a, 'b)\ BR$

12.6 Fixity

Right Infix 300:

$$=_p \quad =_q \quad =_r \quad \leq_{t3} \quad \leq_{t4} \quad \leq_{\in} \quad \lhd_o$$

12.7 Definitions

DPO	$\vdash \exists\ f\bullet\ TypeDefn\ (\lambda\ y\bullet\ T)\ f$
absDPO	
repDPO	$\vdash (\forall\ a\bullet\ absDPO\ (repDPO\ a) = a)$ $\wedge\ (\forall\ r\bullet\ repDPO\ (absDPO\ r) = r)$
dpoB	$\vdash dpoB = absDPO\ (InR\ F)$
dpoT	$\vdash dpoT = absDPO\ (InR\ T)$
dpoE	$\vdash \forall\ e\bullet\ dpoE\ e = absDPO\ (InL\ e)$
dpoV	$\vdash \forall\ x\bullet\ dpoV\ x = OutL\ (repDPO\ x)$
dpoUdef	$\vdash \forall\ x\bullet\ dpoUdef\ x \Leftrightarrow x = dpoB$
dpoOdef	$\vdash \forall\ x\bullet\ dpoOdef\ x \Leftrightarrow x = dpoT$
Dpo	$\vdash \forall\ x\ y\bullet\ Dpo\ x\ y \Leftrightarrow x = y \vee x = dpoB \vee y = dpoT$
pTrue	$\vdash pTrue = Value\ T$
pFalse	$\vdash pFalse = Value\ F$
pU	$\vdash pU = Undefined$
$\leq_{t3}$	$\vdash \forall\ t1\ t2\bullet\ t1 \leq_{t3} t2 \Leftrightarrow t1 = t2 \vee t1 = pU$
fTrue	$\vdash fTrue = dpoE\ T$
fFalse	$\vdash fFalse = dpoE\ F$
fB	$\vdash fB = dpoB$
fT	$\vdash fT = dpoT$
$\leq_{t4}$	$\vdash \forall\ t1\ t2\bullet\ t1 \leq_{t4} t2 \Leftrightarrow t1 = t2 \vee t1 = fB \vee t2 = fT$
CompFTV	$\vdash CompFTV$ $= \{\{\};\ \{fB\};\ \{fFalse\};\ \{fTrue\};\ \{fB;\ fFalse\};$ $\{fB;\ fTrue\}\}$
CoCompFTV	$\vdash CoCompFTV$ $= \{\{\};\ \{fT\};\ \{fFalse\};\ \{fTrue\};\ \{fFalse;\ fT\};$ $\{fTrue;\ fT\}\}$
$\lhd_o$	$\vdash \forall\ V\ r$ $\bullet\ V \lhd_o r$ $= (\lambda\ x\ y\bullet\ if\ x \in V \wedge y \in V\ then\ r\ x\ y\ else\ x = y)$
ConjOrder	$\vdash \forall\ r1\ r2\bullet\ ConjOrder\ r1\ r2 = (\lambda\ x\ y\bullet\ r1\ x\ y \wedge r2\ x\ y)$
DerivedOrder	$\vdash \forall\ f\ r\bullet\ DerivedOrder\ f\ r = (\lambda\ x\ y\bullet\ r\ (f\ x)\ (f\ y))$
Pw	$\vdash \forall\ r\bullet\ Pw\ r = (\lambda\ lo\ ro\bullet\ \forall\ x\bullet\ r\ (lo\ x)\ (ro\ x))$

PwS $\vdash \forall\ V\ r$
$\bullet\ PwS\ V\ r = (\lambda\ lo\ ro \bullet\ \forall\ x \bullet\ x \in V \Rightarrow r\ (lo\ x)\ (ro\ x))$

PrO $\vdash \forall\ ra\ rb$
$\bullet\ PrO\ ra\ rb$
$= (\lambda\ l\ r$
$\bullet\ ra\ (Fst\ l)\ (Fst\ r) \wedge rb\ (Snd\ l)\ (Snd\ r))$

OptO $\vdash \forall\ rl$
$\bullet\ OptO\ rl$
$= (\lambda\ l\ r$
$\bullet\ l = Undefined$
$\vee\ IsDefined\ l$
$\wedge\ IsDefined\ r$
$\wedge\ rl\ (ValueOf\ l)\ (ValueOf\ r))$

DpoEO $\vdash \forall\ rl$
$\bullet\ DpoEO\ rl$
$= (\lambda\ l\ r$
$\bullet\ dpoUdef\ l$
$\vee\ dpoOdef\ r$
$\vee\ (\exists\ le\ re$
$\bullet\ rl\ le\ re \wedge l = dpoE\ le \wedge r = dpoE\ re))$

IxO $\vdash \forall\ r \bullet\ IxO\ r = Pw\ (OptO\ r)$

IxO2 $\vdash \forall\ d\ r$
$\bullet\ IxO2\ (d,\ r)$
$= (\lambda\ x\ y$
$\bullet\ \mathit{if}\ x \in d \wedge y \in d$
$\mathit{then}\ IxDom\ x = IxDom\ y \wedge IxO\ r\ x\ y$
$\mathit{else}\ x = y)$

SetO $\vdash \forall\ r$
$\bullet\ SetO\ r$
$= (\lambda\ m\ n$
$\bullet\ (\forall\ x \bullet\ x \in m \Rightarrow (\exists\ y \bullet\ y \in n \wedge r\ x\ y))$
$\wedge\ (\forall\ y \bullet\ y \in n \Rightarrow (\exists\ x \bullet\ x \in m \wedge r\ x\ y)))$

SetO2 $\vdash \forall\ r$
$\bullet\ SetO2\ r$
$= (\lambda\ m\ n \bullet\ \forall\ x \bullet\ x \in m \Rightarrow (\exists\ y \bullet\ r\ x\ y \wedge y \in n))$

FunSetO $\vdash \forall\ r$
$\bullet\ FunSetO\ r$
$= (\lambda\ s\ t$
$\bullet\ Pw$
$(SetO\ r)$
$(\lambda\ x \bullet\ \{v | \exists\ y \bullet\ y \in s \wedge v = y\ x\})$
$(\lambda\ x \bullet\ \{v | \exists\ y \bullet\ y \in t \wedge v = y\ x\}))$

FunProd $\vdash \forall\ f\ g \bullet\ FunProd\ f\ g = (\lambda\ x \bullet\ (f\ x,\ g\ x))$

FunLeft $\vdash \forall\ f \bullet\ FunLeft\ f = (\lambda\ x \bullet\ (f\ x,\ Snd\ x))$

FunRight $\vdash \forall\ f \bullet\ FunRight\ f = (\lambda\ x \bullet\ (Fst\ x,\ f\ x))$

Pr2BrT $\vdash \forall\ \$=_p \bullet\ Pr2BrT\ \$=_p = (\lambda\ x\ y \bullet\ fTrue \leq_{t4} x =_p y)$

Pr2BrF $\vdash \forall\ \$=_p \bullet\ Pr2BrF\ \$=_p = (\lambda\ x\ y \bullet\ fFalse \leq_{t4} x =_p y)$

BrTF2Pr $\vdash \forall\ \$=_p\ \$=_q$
$\bullet\ BrTF2Pr\ \$=_p\ \$=_q$
$= (\lambda\ x\ y$
$\bullet\ Lub$

$\$\leq_{t4}$
$\{ftv$
$\quad | x =_p y \land ftv = fTrue$
$\qquad \lor x =_p y \land ftv = fFalse\})$

12.8 Theorems

one_one_DPO_lemma
$\vdash OneOne\ repDPO \land OneOne\ absDPO$

dpo_distinct_clauses
$\vdash \neg\ dpoT = dpoB$
$\quad \land \neg\ dpoB = dpoT$
$\quad \land (\forall\ e$
$\quad \bullet \neg\ dpoE\ e = dpoT$
$\qquad \land \neg\ dpoE\ e = dpoB$
$\qquad \land \neg\ dpoT = dpoE\ e$
$\qquad \land \neg\ dpoB = dpoE\ e)$

dpoe_inj_lemma
$\vdash \forall\ e\ f \bullet\ dpoE\ e = dpoE\ f \Leftrightarrow e = f$

dpo_cases_thm
$\vdash \forall\ x \bullet\ x = dpoB \lor x = dpoT \lor (\exists\ e \bullet\ x = dpoE\ e)$

dpove_lemma1 $\vdash \forall\ e \bullet\ dpoV\ (dpoE\ e) = e$

dpodef_lemma1
$\vdash dpoUdef\ dpoB$
$\quad \land dpoOdef\ dpoT$
$\quad \land \neg\ dpoUdef\ dpoT$
$\quad \land \neg\ dpoOdef\ dpoB$
$\quad \land (\forall\ e \bullet \neg\ dpoUdef\ (dpoE\ e) \land \neg\ dpoOdef\ (dpoE\ e))$

dpoev_lemma1 $\vdash \forall\ x \bullet \neg\ dpoUdef\ x \land \neg\ dpoOdef\ x \Rightarrow dpoE\ (dpoV\ x) = x$

dpo_rpou_lemma
$\vdash RpoU\ Dpo$

dpo_glbs_exist_thm
$\vdash GlbsExist\ Dpo$

dpo_lubs_exist_thm
$\vdash LubsExist\ Dpo$

tv_cases_thm $\vdash \forall\ x \bullet\ x = pTrue \lor x = pFalse \lor x = pU$

tv_distinct_clauses
$\vdash \neg\ pTrue = pFalse$
$\quad \land \neg\ pTrue = pU$
$\quad \land \neg\ pFalse = pTrue$
$\quad \land \neg\ pFalse = pU$
$\quad \land \neg\ pU = pTrue$
$\quad \land \neg\ pU = pFalse$

$\leq_{t3}$_refl_thm
$\vdash \forall\ x \bullet\ x \leq_{t3} x$

$\leq_{t3}$_trans_thm
$\vdash \forall\ x\ y\ z \bullet\ x \leq_{t3} y \land y \leq_{t3} z \Rightarrow x \leq_{t3} z$

$\leq_{t3}$_antisym_thm
$\vdash \forall\ x\ y \bullet\ x \leq_{t3} y \land y \leq_{t3} x \Rightarrow x = y$

$\leq_{t3}$_clauses
$\vdash (\forall\ x \bullet\ pU \leq_{t3} x)$

$\land \lnot pTrue \leq_{t3} pU$
$\land \lnot pFalse \leq_{t3} pU$
$\land \lnot pFalse \leq_{t3} pTrue$
$\land \lnot pTrue \leq_{t3} pFalse$

$\leq_{t3}$_partialorder_thm

$\vdash \forall Y \bullet PartialOrder (Y, \$\leq_{t3})$

lin_$\leq_{t3}$_lemma

$\vdash \forall Y$
$\bullet LinearOrder (Y, \$\leq_{t3})$
$\Leftrightarrow \lnot pTrue \in Y \lor \lnot pFalse \in Y$

ccrpou_$\leq_{t3}$_thm

$\vdash CcRpoU \$\leq_{t3}$

ftv_cases_thm

$\vdash \forall x \bullet x = fTrue \lor x = fFalse \lor x = fB \lor x = fT$

ftv_distinct_clauses

$\vdash \lnot fTrue = fFalse$
$\land \lnot fTrue = fB$
$\land \lnot fTrue = fT$
$\land \lnot fFalse = fTrue$
$\land \lnot fFalse = fB$
$\land \lnot fFalse = fT$
$\land \lnot fB = fTrue$
$\land \lnot fB = fFalse$
$\land \lnot fB = fT$
$\land \lnot fT = fTrue$
$\land \lnot fT = fFalse$
$\land \lnot fT = fB$

ftvs_cases_thm

$\vdash \forall x$
$\bullet x = \{\}$
$\lor x = \{fB\}$
$\lor x = \{fFalse\}$
$\lor x = \{fTrue\}$
$\lor x = \{fT\}$
$\lor x = \{fB; fFalse\}$
$\lor x = \{fB; fTrue\}$
$\lor x = \{fB; fT\}$
$\lor x = \{fFalse; fTrue\}$
$\lor x = \{fFalse; fT\}$
$\lor x = \{fTrue; fT\}$
$\lor x = \{fB; fFalse; fTrue\}$
$\lor x = \{fB; fFalse; fT\}$
$\lor x = \{fB; fTrue; fT\}$
$\lor x = \{fFalse; fTrue; fT\}$
$\lor x = \{fB; fFalse; fTrue; fT\}$

$\leq_{t4}$_dpo_thm

$\vdash \$\leq_{t4} = Dpo$

$\leq_{t4}$_refl_thm

$\vdash \forall x \bullet x \leq_{t4} x$

$\leq_{t4}$_trans_thm

$\vdash \forall x\ y\ z \bullet x \leq_{t4} y \land y \leq_{t4} z \Rightarrow x \leq_{t4} z$

$\leq_{t4}$***_antisym_thm***

$\vdash \forall\ x\ y \bullet\ x \leq_{t4} y \land y \leq_{t4} x \Rightarrow x = y$

$\leq_{t4}$***_antisym_thm2***

$\vdash Antisym\ (Universe,\ \$\leq_{t4})$

ft_fb_thm $\vdash \forall\ x \bullet\ (fT \leq_{t4} x \Leftrightarrow x = fT) \land (x \leq_{t4} fB \Leftrightarrow x = fB)$

$\leq_{t4}$***_partialorder_thm***

$\vdash \forall\ Y \bullet\ PartialOrder\ (Y,\ \$\leq_{t4})$

$\leq_{t4}$***_lin_lemma***

$\vdash \forall\ Y$
$\bullet\ LinearOrder\ (Y,\ \$\leq_{t4})$
$\Leftrightarrow \neg\ fTrue \in Y \lor \neg\ fFalse \in Y$

eq_ft_fc_clauses

$\vdash \forall\ x$
$\bullet\ fFalse \leq_{t4} x \land \neg\ x = fFalse$
$\lor fTrue \leq_{t4} x \land \neg\ x = fTrue$
$\lor fT \leq_{t4} x$
$\lor fFalse \leq_{t4} x \land fTrue \leq_{t4} x$
$\Rightarrow x = fT$

eq_fb_fc_clauses

$\vdash \forall\ x$
$\bullet\ x \leq_{t4} fFalse \land \neg\ x = fFalse$
$\lor x \leq_{t4} fTrue \land \neg\ x = fTrue$
$\lor x \leq_{t4} fB$
$\lor x \leq_{t4} fTrue \land x \leq_{t4} fFalse$
$\Rightarrow x = fB$

$\leq_{t4}$***_glbs_exist_thm***

$\vdash GlbsExist\ \$\leq_{t4}$

$\leq_{t4}$***_lubs_exist_thm***

$\vdash LubsExist\ \$\leq_{t4}$

$\leq_{t4}$***_lub_islub_lemma***

$\vdash \forall\ s\ e \bullet\ Lub\ \$\leq_{t4}\ s = e \Leftrightarrow IsLub\ \$\leq_{t4}\ s\ e$

$\leq_{t4}$***_crpou_thm***

$\vdash CRpoU\ \$\leq_{t4}$

$\leq_{t4}$***_lub_clauses***

$\vdash Lub\ \$\leq_{t4}\ \{\} = fB$
$\land Lub\ \$\leq_{t4}\ \{fT\} = fT$
$\land Lub\ \$\leq_{t4}\ \{fTrue\} = fTrue$
$\land Lub\ \$\leq_{t4}\ \{fTrue;\ fT\} = fT$
$\land Lub\ \$\leq_{t4}\ \{fFalse\} = fFalse$
$\land Lub\ \$\leq_{t4}\ \{fFalse;\ fT\} = fT$
$\land Lub\ \$\leq_{t4}\ \{fFalse;\ fTrue\} = fT$
$\land Lub\ \$\leq_{t4}\ \{fFalse;\ fTrue;\ fT\} = fT$
$\land Lub\ \$\leq_{t4}\ \{fB\} = fB$
$\land Lub\ \$\leq_{t4}\ \{fB;\ fT\} = fT$
$\land Lub\ \$\leq_{t4}\ \{fB;\ fTrue\} = fTrue$
$\land Lub\ \$\leq_{t4}\ \{fB;\ fTrue;\ fT\} = fT$
$\land Lub\ \$\leq_{t4}\ \{fB;\ fFalse\} = fFalse$
$\land Lub\ \$\leq_{t4}\ \{fB;\ fFalse;\ fT\} = fT$
$\land Lub\ \$\leq_{t4}\ \{fB;\ fFalse;\ fTrue\} = fT$
$\land Lub\ \$\leq_{t4}\ \{fB;\ fFalse;\ fTrue;\ fT\} = fT$

$\leq_{t4}$***_lub_thm***

$\vdash \forall\ s$
$\bullet\ Lub\ \$\leq_{t4}\ s$
$= (if\ fT \in s$
$then\ fT$
$else\ if\ fTrue \in s$
$then\ if\ fFalse \in s\ then\ fT\ else\ fTrue$
$else\ if\ fFalse \in s$
$then\ fFalse$
$else\ fB)$

$\leq_{t4}$_glb_clauses

$\vdash Glb\ \$\leq_{t4}\ \{\} = fT$
$\land Glb\ \$\leq_{t4}\ \{fT\} = fT$
$\land Glb\ \$\leq_{t4}\ \{fTrue\} = fTrue$
$\land Glb\ \$\leq_{t4}\ \{fTrue;\ fT\} = fTrue$
$\land Glb\ \$\leq_{t4}\ \{fFalse\} = fFalse$
$\land Glb\ \$\leq_{t4}\ \{fFalse;\ fT\} = fFalse$
$\land Glb\ \$\leq_{t4}\ \{fFalse;\ fTrue\} = fB$
$\land Glb\ \$\leq_{t4}\ \{fFalse;\ fTrue;\ fT\} = fB$
$\land Glb\ \$\leq_{t4}\ \{fB\} = fB$
$\land Glb\ \$\leq_{t4}\ \{fB;\ fT\} = fB$
$\land Glb\ \$\leq_{t4}\ \{fB;\ fTrue\} = fB$
$\land Glb\ \$\leq_{t4}\ \{fB;\ fTrue;\ fT\} = fB$
$\land Glb\ \$\leq_{t4}\ \{fB;\ fFalse\} = fB$
$\land Glb\ \$\leq_{t4}\ \{fB;\ fFalse;\ fT\} = fB$
$\land Glb\ \$\leq_{t4}\ \{fB;\ fFalse;\ fTrue\} = fB$
$\land Glb\ \$\leq_{t4}\ \{fB;\ fFalse;\ fTrue;\ fT\} = fB$

compftv_lemma

$\vdash \forall\ s$
$\bullet\ s \in CompFTV$
$\Leftrightarrow \neg\ fT \in s \land (\neg\ fTrue \in s \lor \neg\ fFalse \in s)$

cocompftv_lemma

$\vdash \forall\ s$
$\bullet\ s \in CoCompFTV$
$\Leftrightarrow \neg\ fB \in s \land (\neg\ fTrue \in s \lor \neg\ fFalse \in s)$

CompFTV_Lub_lemma

$\vdash \forall\ s \bullet\ s \in CompFTV \Leftrightarrow \neg\ Lub\ \$\leq_{t4}\ s = fT$

Lub_CompFTV_lemma

$\vdash \forall\ s \bullet\ Lub\ \$\leq_{t4}\ s = fT \Leftrightarrow \neg\ s \in CompFTV$

CoCompFTV_Glb_lemma

$\vdash \forall\ s \bullet\ s \in CoCompFTV \Leftrightarrow \neg\ Glb\ \$\leq_{t4}\ s = fB$

Glb_CoCompFTV_lemma

$\vdash \forall\ s \bullet\ Glb\ \$\leq_{t4}\ s = fB \Leftrightarrow \neg\ s \in CoCompFTV$

$\leq_{t4}$_lin_lub_lemma

$\vdash \forall\ X$
$\bullet\ LinearOrder\ (X,\ \$\leq_{t4})$
$\Rightarrow (fT \leq_{t4} Lub\ \$\leq_{t4}\ X \Leftrightarrow fT \in X)$

$\leq_{t4}$_lin_lub_lemma2

$\vdash \forall\ X$
$\bullet\ LinearOrder\ (X,\ \$\leq_{t4})$
$\Rightarrow (fT = Lub\ \$\leq_{t4}\ X \Leftrightarrow fT \in X)$

$\leq_{t4}$_lin_glb_lemma

$\vdash \forall\ X$
$\bullet\ LinearOrder\ (X,\ \$\leq_{t4})$
$\Rightarrow (Glb\ \$\leq_{t4}\ X \leq_{t4} fB \Leftrightarrow fB \in X)$

$\leq_{t4}$***_lin_glb_lemma2***
$\vdash \forall\ X$
$\bullet\ LinearOrder\ (X,\ \$\leq_{t4})$
$\Rightarrow (Glb\ \$\leq_{t4}\ X = fB \Leftrightarrow fB \in X)$

do_lubs_exist_thm
$\vdash \forall\ f\ r$
$\bullet\ LubsExist\ r \land Onto\ f \Rightarrow LubsExist\ (DerivedOrder\ f\ r)$

do_glbs_exist_thm
$\vdash \forall\ f\ r$
$\bullet\ GlbsExist\ r \land Onto\ f \Rightarrow GlbsExist\ (DerivedOrder\ f\ r)$

wf_derived_order_thm
$\vdash \forall\ r$
$\bullet\ well\_founded\ r$
$\Rightarrow (\forall\ f \bullet\ well\_founded\ (DerivedOrder\ f\ r))$

lubsexist_dofst_thm
$\vdash \forall\ f\ r \bullet\ LubsExist\ r \Rightarrow LubsExist\ (DerivedOrder\ Fst\ r)$

glbsexist_dofst_thm
$\vdash \forall\ f\ r \bullet\ GlbsExist\ r \Rightarrow GlbsExist\ (DerivedOrder\ Fst\ r)$

lubsexist_dosnd_thm
$\vdash \forall\ f\ r \bullet\ LubsExist\ r \Rightarrow LubsExist\ (DerivedOrder\ Snd\ r)$

glbsexist_dosnd_thm
$\vdash \forall\ f\ r \bullet\ GlbsExist\ r \Rightarrow GlbsExist\ (DerivedOrder\ Snd\ r)$

pw_ccrpou_thm
$\vdash \forall\ r \bullet\ CcRpoU\ r \Rightarrow CcRpoU\ (Pw\ r)$

pw_lubs_exist_thm
$\vdash \forall\ r \bullet\ LubsExist\ r \Rightarrow LubsExist\ (Pw\ r)$

pw_glbs_exist_thm
$\vdash \forall\ r \bullet\ GlbsExist\ r \Rightarrow GlbsExist\ (Pw\ r)$

pw_crpou_thm $\vdash \forall\ r \bullet\ CRpoU\ r \Rightarrow CRpoU\ (Pw\ r)$

pw_$\leq_{t4}$***_lubsexist_thm***
$\vdash LubsExist\ (Pw\ \$\leq_{t4})$

pw_$\leq_{t4}$***_glbsexist_thm***
$\vdash GlbsExist\ (Pw\ \$\leq_{t4})$

pw_$\leq_{t4}$***_crpou_thm***
$\vdash CRpoU\ (Pw\ \$\leq_{t4})$

pws_lubs_exist_thm
$\vdash \forall\ V\ r \bullet\ LubsExist\ r \Rightarrow LubsExist\ (PwS\ V\ r)$

pws_glbs_exist_thm
$\vdash \forall\ V\ r \bullet\ GlbsExist\ r \Rightarrow GlbsExist\ (PwS\ V\ r)$

pro_refl_lemma
$\vdash \forall\ ra\ rb$
$\bullet\ Refl\ (Universe,\ ra) \land Refl\ (Universe,\ rb)$
$\Rightarrow Refl\ (Universe,\ PrO\ ra\ rb)$

pro_partialorder_lemma
$\vdash \forall\ ra\ rb$
$\bullet\ PartialOrder\ (Universe,\ ra)$
$\land\ PartialOrder\ (Universe,\ rb)$
$\Rightarrow PartialOrder\ (Universe,\ PrO\ ra\ rb)$

pro_rpo_lemma

$\vdash \forall\ ra\ rb$
$\quad \bullet\ Rpo\ (Universe,\ ra) \land Rpo\ (Universe,\ rb)$
$\quad\quad \Rightarrow Rpo\ (Universe,\ PrO\ ra\ rb)$

pro_lubs_exist_thm

$\vdash \forall\ ra\ rb$
$\quad \bullet\ LubsExist\ ra \land LubsExist\ rb \Rightarrow LubsExist\ (PrO\ ra\ rb)$

pro_glbs_exist_thm

$\vdash \forall\ ra\ rb$
$\quad \bullet\ GlbsExist\ ra \land GlbsExist\ rb \Rightarrow GlbsExist\ (PrO\ ra\ rb)$

pro_crpou_thm

$\vdash \forall\ ra\ rb \bullet\ CRpoU\ ra \land CRpoU\ rb \Rightarrow CRpoU\ (PrO\ ra\ rb)$

dpoeo_lubs_exist_thm

$\vdash \forall\ r \bullet\ LubsExist\ r \Rightarrow LubsExist\ (DpoEO\ r)$

dpoeo_glbs_exist_thm

$\vdash \forall\ r \bullet\ GlbsExist\ r \Rightarrow GlbsExist\ (DpoEO\ r)$

$\leq_{t4}$_trich_lub_ft_lemma

$\vdash \forall\ X\ z$
$\quad \bullet\ Trich\ (X,\ \$\leq_{t4}) \land fT \leq_{t4} Lub\ \$\leq_{t4}\ X \Rightarrow fT \in X$

lub_increasing_lemma

$\vdash \forall\ r$
$\quad \bullet\ RpoU\ r \land LubsExist\ r$
$\quad\quad \Rightarrow Increasing\ (SetO\ r)\ r\ (Lub\ r)$

lub_increasing2_lemma

$\vdash \forall\ r$
$\quad \bullet\ RpoU\ r \land LubsExist\ r$
$\quad\quad \Rightarrow Increasing\ (SetO2\ r)\ r\ (Lub\ r)$

lub_increasing_lemma2

$\vdash \forall\ r \bullet\ CRpoU\ r \Rightarrow Increasing\ (SetO\ r)\ r\ (Lub\ r)$

lub_increasing2_lemma2

$\vdash \forall\ r \bullet\ CRpoU\ r \Rightarrow Increasing\ (SetO2\ r)\ r\ (Lub\ r)$

lub_increasing_lemma3

$\vdash \forall\ r$
$\quad \bullet\ CRpoU\ r$
$\quad\quad \Rightarrow (\forall\ x\ y \bullet\ SetO\ r\ x\ y \Rightarrow r\ (Lub\ r\ x)\ (Lub\ r\ y))$

lub_increasing2_lemma3

$\vdash \forall\ r$
$\quad \bullet\ CRpoU\ r$
$\quad\quad \Rightarrow (\forall\ x\ y \bullet\ SetO2\ r\ x\ y \Rightarrow r\ (Lub\ r\ x)\ (Lub\ r\ y))$

funprod_increasing_thm

$\vdash \forall\ f\ g\ ra\ rb\ rc$
$\quad \bullet\ Increasing\ ra\ rb\ f \land Increasing\ ra\ rc\ g$
$\quad\quad \Rightarrow Increasing\ ra\ (PrO\ rb\ rc)\ (FunProd\ f\ g)$

funleft_increasing_thm

$\vdash \forall\ f\ ra\ rb\ rc$
$\quad \bullet\ Increasing\ (PrO\ ra\ rc)\ rb\ f$
$\quad\quad \Rightarrow Increasing\ (PrO\ ra\ rc)\ (PrO\ rb\ rc)\ (FunLeft\ f)$

funright_increasing_thm

$\vdash \forall\ f\ ra\ rb\ rc$
$\quad \bullet\ Increasing\ (PrO\ rc\ ra)\ rb\ f$
$\quad\quad \Rightarrow Increasing\ (PrO\ rc\ ra)\ (PrO\ rc\ rb)\ (FunRight\ f)$

linear_funimage_thm

$\vdash \forall\ r1\ \ r2\ \ f\ \ X$

$\bullet\ Increasing\ \ r1\ \ r2\ \ f$

$\quad \wedge\ LinearOrder\ (X,\ r1)$

$\quad \wedge\ PartialOrder\ (FunImage\ f\ \ X,\ r2)$

$\Rightarrow\ LinearOrder\ (FunImage\ f\ \ X,\ r2)$

13 The Theory misc2

13.1 Parents

GS *misc1*

13.2 Children

ifol *t001b t001a*

13.3 Constants

FunImage$_g$	$(GS \to 'a) \to GS \to 'a\ \mathbb{P}$
IsVal	$'a\ IS \to GS \to 'a$
IsRan	$'a\ IS \to 'a\ \mathbb{P}$
IsDom	$'a\ IS \to GS\ \mathbb{P}$
IsSDom	$'a\ IS \to GS\ \mathbb{P}$
IsOd	$'a\ IS \to GS\ \mathbb{P}$
IsUd	$'a\ IS \to GS\ \mathbb{P}$
IsTDom	$'a\ IS \to GS\ \mathbb{P} \times GS\ \mathbb{P} \times GS\ \mathbb{P}$
IsOverRide	$('a\ IS, 'a\ IS)\ BR$
IsEO	$('a, BOOL)\ BR \to ('a\ IS, BOOL)\ BR$
$\$\leq_\in$	$((GS, FTV)\ BR, BOOL)\ BR$

13.4 Type Abbreviations

$'a$ ***IS***	$'a\ IS$

13.5 Definitions

FunImage$_g$	$\vdash \forall\ f\ s \bullet\ FunImage_g\ f\ s = \{x \mid \exists\ y \bullet\ y \in_g s \land x = f\ y\}$
IsVal	$\vdash \forall\ is\ g \bullet\ IsVal\ is\ g = dpoV\ (is\ g)$
IsRan	$\vdash \forall\ is \bullet\ IsRan\ is = \{v \mid \exists\ \alpha \bullet\ dpoE\ v = is\ \alpha\}$
IsDom	$\vdash \forall\ is \bullet\ IsDom\ is = \{i \mid \neg\ is\ i = dpoB\}$
IsSDom	$\vdash \forall\ is \bullet\ IsSDom\ is = \{i \mid \neg\ (is\ i = dpoB \lor is\ i = dpoT)\}$
IsOd	$\vdash \forall\ is \bullet\ IsOd\ is = \{i \mid is\ i = dpoT\}$
IsUd	$\vdash \forall\ is \bullet\ IsUd\ is = \{i \mid is\ i = dpoB\}$
IsTDom	$\vdash \forall\ is \bullet\ IsTDom\ is = (IsSDom\ is, IsUd\ is, IsOd\ is)$
IsOverRide	$\vdash \forall\ is1\ is2$ $\bullet\ IsOverRide\ is1\ is2$ $= (\lambda\ i$ $\bullet\ if\ \neg\ dpoUdef\ (is2\ i)\ then\ is2\ i\ else\ is1\ i)$
IsEO	$\vdash \forall\ r \bullet\ IsEO\ r = Pw\ (DpoEO\ r)$
$\leq_\in$	$\vdash \$\leq_\in = Pw\ (Pw\ \$\leq_{t4})$

13.6 Theorems

funimage$_g$_fc_lemma

$\vdash \forall\ f\ s\ x \bullet\ x \in_g s \Rightarrow f\ x \in FunImage_g\ f\ s$

is_domran_lemma

$\vdash \forall\ x\ y \bullet\ x \in IsSDom\ y \Rightarrow IsVal\ y\ x \in IsRan\ y$

istdom_eq_fc_lemma

$\vdash \forall\ x\ y \bullet\ IsTDom\ x = IsTDom\ y \Rightarrow IsUd\ x = IsUd\ y \land IsOd\ x = IsOd\ y \land IsSDom\ x = IsSDom\ y \land IsDom\ x = IsDom\ y$

isoverride_isdom_lemma

$\vdash \forall\ x\ y \bullet\ IsDom\ (IsOverRide\ x\ y) = IsDom\ x \cup IsDom\ y$

isoverride_isud_lemma

$\vdash \forall\ x\ y \bullet\ IsUd\ (IsOverRide\ x\ y) = IsUd\ x \setminus IsDom\ y$

isoverride_isod_lemma

$\vdash \forall\ x\ y \bullet\ IsOd\ (IsOverRide\ x\ y) = IsOd\ y \cup IsOd\ x \setminus IsDom\ y$

isoverride_issdom_lemma

$\vdash \forall\ x\ y \bullet\ IsSDom\ (IsOverRide\ x\ y) = IsSDom\ y \cup IsSDom\ x \setminus IsDom\ y$

is_lubs_exist_thm

$\vdash \forall\ r \bullet\ LubsExist\ r \Rightarrow LubsExist\ (IsEO\ r)$

is_glbs_exist_thm

$\vdash \forall\ r \bullet\ GlbsExist\ r \Rightarrow GlbsExist\ (IsEO\ r)$

crpou_$\leq_\in$_thm

$\vdash CRpoU\ \$\leq_\in$

ccrpou_$\leq_\in$_thm

$\vdash CcRpoU\ \$\leq_\in$

$\leq_\in$_clauses $\vdash GlbsExist\ \$\leq_\in \land LubsExist\ \$\leq_\in \land (\forall\ x \bullet\ x \leq_\in x)$

$\leq_\in$_fc_clauses

$\vdash (\forall\ x\ y\ z \bullet\ x \leq_\in y \land y \leq_\in z \Rightarrow x \leq_\in z) \land (\forall\ x\ y \bullet\ x \leq_\in y \land y \leq_\in x \Rightarrow x = y)$

$\leq_\in$_lub_thm $\vdash \forall\ G \bullet\ Lub\ \$\leq_\in G = (\lambda\ x\ y \bullet\ Lub\ \$\leq_{t4}\ \{w | \exists\ g \bullet\ g \in G \land w = g\ x\ y\})$

$\leq_\in$_increasing_pointwise_thm

$\vdash Increasing\ \$\leq_\in\ \$\leq_{t4}\ (\lambda\ f \bullet\ f\ x'\ y)$

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