PolySets

foundational ontologies for formal mathematics

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Abstract

Pragmatic considerations arising in the formalisation of mathematics (using tools such as Isabelle, HOL4, ProofPower ...) can be seen to motivate the adoption of non-well-founded foundational ontologies. These considerations also yield intuitions about the kinds of non-well-founded sets which are desirable. Inductively defined classes of well-founded sets may be used to represent a domain which includes both the well-founded sets and the desired non-well-founded sets.

By such means we arrive at PolySets. Some of the motives and intuitions will be described, together with details of the construction of the PolySets.

The PolySets appear to meet those desiderata most closely related to the intuitions on which they were based, but not all the desiderata. The methods used can be strengthened and generalised providing a framework in which one might speak of "the open ended nature of non-well-founded sets" and in which the incoherent ideal of full infinitary comprehension can be approached. This more general "framework" will be sketched.

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A Semantic Stack

Illative Lambda Calculus / Type Theory
Non-well-founded ontology
Well-founded ontology

The following "semantic" gives a picture of the enterprise of which the presented work is a part. This consists in the design of a language for the formal development of mathematics, something like an illative lambda calculus (i,e, a type free lambda calculus strengthened to yield a foundation for mathematics) and a type theoretic foundation system. The idea is to address certain pragmatic issues which arise in foundation systems from the constraint to a semantics which is well-founded. This requires adopting a semantics which is not well-founded.

The best places to start in constructing a rich non-well-founded ontology is set theory, and the best approaches build on the less controversial domain of well-founded sets.

We therefore seek to construct from a well-founded set theoretic ontology a non-well-founded ontology using intuitions derived from pragmatic problems arising in the mechanisation of formal mathematics.

Outline

Contents

1 The Motivation for PolySets

Motivation

- to provide an ontology suitable for mechanised formal mathematics
- specifically to enable ML-like local polymorphic definitions in HOL-like languages.
- also to support *locales* or other specification structuring features
- possibly to allow improved formal treatment of abstract mathematics such as category theory.
- to leave concrete mathematics (numbers, analysis, geometry) untouched.

Let in ML - types

The simplest of polymorphic functions:

SML
```
fun id x = x;
```

```
val id = fn : 'a -> 'a
```

SML
```
let fun id x = x in (id id) (id 0) end;
```

```
val it = 0 : int
```

Let in HOL - types

HOL Constant

$\boldsymbol{id} : {}'a \to {}'a$

$\forall x \bullet\ id\ x = x$

$rewrite\_conv\ [get\_spec\ \ulcorner id \urcorner]\ \ulcorner (id\ id)\ (id\ 0) \urcorner$;

$val\ it = \vdash id\ id\ (id\ 0) = 0 : THM$

$\ulcorner let\ id\ x = x\ in\ (id\ id)\ (id\ 0) \urcorner$;

$Type\ error\ in\ \ulcorner id\ id \urcorner$
$In\ \ulcorner f\ a \urcorner\ where\ \ulcorner a{:}\sigma \urcorner,\ \ulcorner f \urcorner\ must\ have\ type\ of\ the\ form\ \sigma \to \tau$
$Cannot\ apply\ \ulcorner id{:}(\mathbb{N} \to \mathbb{N}) \urcorner$
$to\ \ulcorner id{:}(\mathbb{N} \to \mathbb{N}) \urcorner$

Semantics and Ontology
in ML:

Semantics *id* denotes a function with the extension: $\{(x, y)|x = y\}$

Just like $(or \lambda x.x)$ in the type-free lambda calculus.

Ontology ML semantics uses reflexive domains with continuous function spaces. This means you can have self-application of functions.

in HOL:

Semantics In HOL *id* denotes family of functions with extension: $\{(X, f)|f = \{(x, x)|x \in X\}\}$

more like the lambda expression: $(\lambda X.\lambda x : X.x)$

Ontology

- the ontology is well-founded
- $\{(x, y)|x = y\}(or \lambda x.x)$ *doesn't exist*
- therefore, make do with families of functions

Strachey/Scott Maxim Ontology first, then semantics, concrete syntax last.

The difference in the type systems of ML and HOL may be attributed to a difference in the semantics of polymorphic functions.

In ML (according to Milner's paper on polymorphism) a polymorphic function is a single function which has many types (one for each instance of the poly-type. In HOL a polymorphic function is a family of monomorphic functions.

2 The Intuitions Behind PolySets

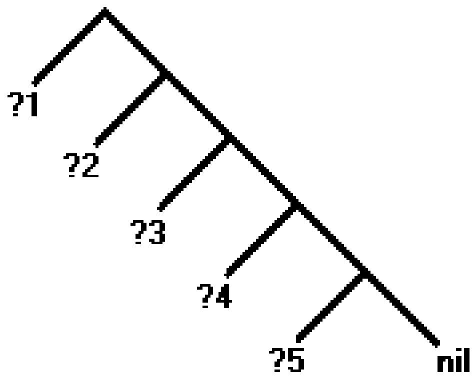
Intuitions I

- Because the ontology underlying HOL is well-founded it lacks the values which polymorphic functions in ML denote.
- Can we contruct an ontology which includes both a full collection of well-founded sets, and those non-well-founded sets which are the graphs of polymorphic functions?

These functions are easily implemented in functional programming languages, so we can ask the question:

- Why is it possible to implement polymorphic functions?

A List



SML

```
fun length nil = 0
|   length (h::t) = (length t) + 1;
```

```
val length = fn : 'a list -> int
```

Intuitions II

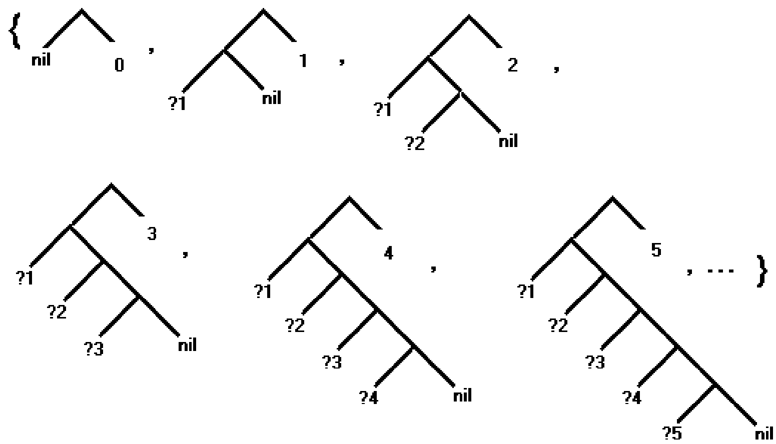
So, the reason why we can implement polymorphic functions is:

- Because polymorphic functions never do anything with the variably typed constutients of their arguments except (possibly) copy them.
- Polymorphic functions only look at the superficial structure of their arguments.

From which we can infer (or conjecture) that:

- The graph of a polymorphic function can be represented by a set of patterns.

The polymorphic *length* function



3 The Construction of PolySets

Constructing the PolySets

We define (using a higher order set theory):

- a class of well-founded sets (the PolySet reps)
- an injection of the well founded sets into the PolySet reps
- an operation of instantiation over sets interpreted as patterns
- a membership relation over the PolySet reps
- an equivalence relation which makes membership extensional

not quite in that order.

The injection of WF into the PolySet reps

(x, y) is the ordered pair of x and y. Let WF be $V(\alpha)$ for some reasonably large ordinal α (say, a Mahlo cardinal).

$$psi(\{\}) = \{\}$$

for non-empty *x*:

$$psi(x) = (\{\}, psi``x)$$

Let $PsOn$ be the image under psi of the (Von Neumann) ordinals in WF:

$$PsOn = psi``Ord$$

The PolySet reps

The class of $PolySet\ reps$ is then defined recursively as:

$$PolySetR = \{(o, s) \mid o \in PsOn \wedge s \in WF \wedge s \subseteq PolySetR\} \cup \{\}$$

In the above:

- "o" is a set of PolySet ordinals to be used as $variables$
- "s" is a set of patterns in which the variables are like wild-cards
- the semantics of the patterns is given by defining instantiation

Instantiation

A *variable assignment* is a $PolySet\ rep$ valued function defined over an initial segment of $PsOn$. An *instance* of a $PolySet\ rep$ is obtained by substituting, for the variables in it, values from a variable assignment. Instantiation is defined as follows, in which '$+$' is addition over $PsOn$.

$$if\ ps \in dom\ va$$
$$then\ inst\ va\ ps = va\ ps$$

$$else\ if\ ps \in PsOn \setminus dom\ va$$
$$then\ inst\ va\ ps = \epsilon ps' \bullet\ (dom\ va + ps' = ps)$$

$$else\ if\ ps \in PolySetR \setminus PsOn$$
$$then\ ps\ has\ the\ form\ (o,\ pss)$$
$$and\ inst\ va\ (o,\ pss) = (o,\ (inst\ va)\ ``\ pss)$$

Membership

Membership over $PolySet\ reps$ is then defined:

$$psr \notin \{\}$$
$$\wedge$$
$$psr \in (o,\ psrs) \Leftrightarrow$$
$$\exists va\ psr2 \bullet\ (domain\ va = o$$
$$\wedge\ psr2 \in psrs$$
$$\wedge\ psr = inst\ va\ psr2)$$

Extensional Equality

extsube :: "(Set × Set)set ⇒ (Set × Set)set"
"extsube e == {(s,t). s ∈ polysets ∧ t ∈ polysets
∧ (∀u. ps_mem u s ⟶ ps_mem u t ∨ (∃v. (u, v) ∈ e ∧ ps_mem v t))}"

exteqe :: "(Set × Set)set ⇒ (Set × Set)set"
"exteqe e == ((extsube e ∩ converse (extsube e)) ∪ (e ∩ converse e))
∩ polysets × polysets"

tcexteq :: "(Set × Set)set ⇒ (Set × Set)set"
"tcexteq e == trancl(exteqe e)"

consts
ps_equiv :: "(Set * Set)set"
inductive
ps_equiv
intros
pseI: "(s,t) ∈ tcexteq ps_equiv ⟹ (s,t) ∈ ps_equiv"
monos tcexteq_mono

Extensional Membership

constdefs
ps_eqc :: "Set set set"
"ps_eqc == polysets // ps_equiv"

pseqc_mem :: "Set set ⇒ Set set ⇒ bool"
"pseqc_mem s t == ∃v w. v ∈ s ∧ w ∈ t ∧ ps_mem v w"

4 Evaluation

Evaluation of PolySets

- Higher Order "axioms"
- Other Properties
- Comparison with CO constructions
- Polymorphism and Locales
- Review of methods

Higher Order Axioms

It is natural to expect that this set theory can be axiomatised with three axioms as follows. There are two primitive constants, a binary membership relation and the predicate “Low” (whose negation is “High”).

- Full extensionality.
- Every set is a member of a galaxy.

  A galaxy is a low set which is closed under:

  low transitive All low members of a galaxy are subsets of the galaxy.

  low power set All the subsets of a low set are low sets and are collected in a low set.

  low sumset All low sets have sumsets, low if all the members of the set are low, otherwise high.

  low replacement The image of a low set under a many-one relation is a low set.

- PolySet abstraction. Any low set together with an ordinal determines a high set whose members are those sets which can be obtained by instantiating a member of the first set according to valuation which is a family of sets indexed by the ordinal.

Other Properties

- Few complements (V may be the only one).
- No essences.
- No Frege cardinals (stick to the Von Neumann ordinals and the alephs).
- No gratuitous failures of $\in$ foundation.
  - All high PolySets have the same size and are larger (externally) than any low PolySet.
  - The only self-membered set is V.
  - All $\in$ loops involve at least one high PolySet.
  - $\in$ restricted to sets smaller than V (i.e. to low sets) is well-founded.

Similarities with CO constructions

Low The “low” PolySets are those in which there are no free variables.

11 There will be something analogous to low comprehension. Any set of PolySets is a low PolySet. High PolySets are all classes of PolySets.

12 The set of low PolySets is not a PolySet (not even a high one).

13 An image of a low PolySet is low, subsets of low PolySets are low.

14 A low PolySet has a low power set.

15 Low PolySets have sumsets. low PolySets of low PolySets have low sumsets.

30 H_{low} is isomorphic to the original universe(?)

32 A canonical injection from the original universe has been defined.

34 The well-founded PolySets are not a PolySet..

Polymorphism and Locales

- I don't know whether I caught all the polymorphic functions.
- A locale is a bit like a let clause:

$$let\ \ sig\ \mid\ pred\ \ in\ \ spec$$

  - Its natural to think of this as functor, but the domain is too large
  - palliatives similar to polymorphism, but not ideal
  - Locales ideally would require full comprehension (?)

Review of Method

- first cut
  - code up representatives for the desired sets (as an inductively defined class of well-founded sets)
  - define membership over these representatives (this might not be easy, or possible!)
  - take a quotient to extensionalise (take the smallest equivalence relation which makes membership extensional and redefine membership over the equivalence classes, this might mess things up)
- this is not so general as it looks
- it does strictly subsume the Church-Oswald constructions
- however, if representatives are not unique then usually membership can only be defined mutually with equality, hence:
- second cut
  - code up representatives for the desired sets
  - mutually define membership and equality
- this really is pretty general, and
- it may be straightforward to give a recursive "definition" ...
- but hard to prove that it has a fixed point.

Going for Full Comprehension

- go for broke...
  - why not code up "the whole lot" at once? (i.e.: all the properties definable in infinitary 1st order set theory)
  - define semantics as a functor F transforming $(\in\ , =)$ pairs (parameterised by V, arbitrary subsets of R the set if codings)
  - then look for subsets S of R such that (F S) has a fixed point
- conjecture: every consistent 1st order set theory has a model of this kind.
- this gives us a "sandbox" in which to experiment with non-well-founded ontologies
- we could ask questions like:
  - when are two ontologies compatible?
  - what notions of maximal ontology make sense?

5 The Iterative Conception of Set

This phrase normally refers to the cumulative heirarchy of well-founded sets. In a recent paper, Thomas Forster has argued that it also embraces well-founded sets, and gives a nice explanation of Church-Oswald constructions of non-well-founded interpretations of set theory showing how they fit in with his view on the iterative conception.

In the course of preparing this talk (not long after seeing Forster's paper) I came to the view that the more general methods I was considering (to realise a richer ontology than the PolySets), which are also I believe, more general than the Church-Oswald constructions, can also be argued as belonging to an iterative conception.

The Iterative Conception of Set

- invented to give an account of the domain of well-founded set theory
- Forster argues that it applies equally to Church-Oswald non-well-founded interpretations of set theory
- Can also be applied to the more general methods under consideration, and arguably to all sets, well-founded or otherwise.

Some Inductive Definitions

- well-founded sets:

  a well-founded set is a definite collection of well-founded sets

- a simple Church-Oswald construction:

  a set is either:

  1. a definite collection of sets, or
  2. the complement of a definite collection of sets

- complemented PolySets:

  a set is either:

  1. a PolySet abstraction, or
  2. the complement of a PolySet abstraction

- infinitarily definable sets:

  a set is either:

  1. a dependently restricted PolySet abstraction, or
  2. the complement of a dependently restricted PolySet abstraction

Defining Truth for Set Theory

Here is a definition of the truth conditions for sentences in first order set theory.

Well-Founded:

- an interpretation of set theory is a transitive well-founded set
- a sentence is false if the set of interpretations in which it is true is bounded
- a sentence is true if the set of interpretations in which it is false is bounded

This has some disadvantages (e.g. that some sentences lack truth values) but has the merit (if it accepted that the cumulative heirarchy cannot be completed) of being simple and natural and of fixing the truth value of most of the sentences that one needs to be definite. e.g. the continuum hypothesis, and large cardinal axioms. It doesn't make the axioms of ZFC true, so that will suffice to put many people off it, but it does make it true that there are standard models of ZFC. It is good for a set theory which is good as an ultimate foundation for abstract semantics.

Non-Well-Founded

- can we produce an analogous account in terms of infinitarily definable properties
- an interpretation is a triple (V, =, $\in$) which has a unique fixed point under the semantics.
- will similar definitions of $true$ and $false$ work?
- probably not, we will have to work harder than that

Finding Maximal Fixed Points

- Lets consider these definable properties as proto-sets and ask: which of the proto-sets can (or should) be sets?
- ideally (after Hilbert): "the consistent ones", but:
  - its not obvious what the relevant notion of consistency is here
  - there may be no notion of consistency such that the collection of consistent sets is a fixed point of our functor

Consistency

- If the semantic functor is coded over *partial relations* then it can be made monotonic, and for any fixed subdomain of XPS it will have a fixed point
- Take the least fixed point over the whole of XPS.
- If this is total we have an interpretation of set theory.
- If not take the subset D of XPS over which it is total, and find a new least fixed point.
- Repeat this process untill a fixed point is obtained which is total.
- The subset of XPS we might call the "consistently non-well-founded sets", and they give an interpretation for first order set theory.

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