

Semantic Foundations for Deductive Methods

delineating the scope of deductive reason

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Abstract. The scope of deductive reason is considered. First a connection is discussed between the scope of sound deductive inference and the notion of set theoretic truth via the concepts *demonstrative* and *analytic*. Then the problem of determining the meaning of set theory and the extension of set theoretic truth is addressed.

1. Introduction

I propose to consider here certain problems in the foundations of abstract semantics. The motivation for this lies in the notions of deductive reasoning, of analyticity and entailment. Set theory is of importance not merely as branch of mathematics, nor even as a foundation for mathematics, but as a foundation for abstract semantics, and thereby as an aid in delimiting the scope of deductive reasoning.

The early parts of the essay provide some considerations supporting this conception of the importance of set theory. The main body of the essay then considers ways in which the semantics of set theory and the extension of set theoretic truth can be made definite.

Though people do engage in deduction in natural languages, such languages will not be considered here. My primary reason for omitting specific consideration of natural languages is that I know no way of including them in the scope of this discussion which would be sufficiently precise and straightforward for present purposes. However, any natural language which has a well defined semantics (even if incomplete) and a deductive system sound with respect to that semantics (even if these exist but are not known) will fall within the scope of the discussion.

It is suggested that insofar as the delineation of the scope of deductive reason is concerned, this can be done without loss of generality by consideration exclusively of formal deductive systems.

2. Demonstrative and Analytic Truth

My aim here is to connect normal practice in establishing the soundness of formal deductive systems with the philosophical concept of analyticity.

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This is done via an argument that the concepts *demonstrative* and *analytic*, suitably defined, are coextensive, the concept demonstrative being defined for this purpose as *derivable in a sound deductive system*.

2.1. DEMONSTRATIVE TRUTH

2.1.1. *Prior Use*

In Aristotle the terms demonstrative and dialectical are used to distinguish two kinds of premises to syllogistic proofs and to distinguish proofs having these kinds of premises.

A demonstrative premise is one “obtained through the first principles of its science”. A dialectical premise is one adopted for argumentative purposes. A demonstrative premise must be true, whereas a dialectical premise need not be.

Furthermore, he says “Demonstrative knowledge must rest on necessary basic truths; for the object of scientific knowledge cannot be other than it is.”, and that “demonstrative truth must be knowledge of a necessary nexus”.

For Aristotle then, demonstrative truths are necessary, because reached by syllogistic reasoning from necessary premises.

In Locke the term demonstrative is reserved for the conclusions of proofs, premises are described as *intuitive*:

For if we will reflect on our own ways of thinking, we will find, that sometimes the mind perceives the agreement or disagreement of two ideas immediately by themselves, without the intervention of any other: and this I think we may call intuitive knowledge.

The wording here is more suggestive of analyticity than of necessity. Demonstrative knowledge is now defined again via the notion of intuition rather than by reference to the syllogism:

Now, in every step reason makes in demonstrative knowledge, there is an intuitive knowledge of that agreement or disagreement it seeks with the next intermediate idea which it uses as a proof...

and the notion of “intuitive” remains here as strong as analytic or necessary.

2.1.2. *Definition*

The standards of modern logic allow what may be thought of as essentially the same concept to be rendered with a greater precision. Now, the achievement of the highest levels of certainty, which Locke attributes to intuitive and demonstrative knowledge, are associated with the theorems of formal deductive systems. The role played by

Locke's concept *intuitive* in giving us the necessary confidence and assurance is supplanted in modern logic by a proof of soundness, conducted about a formal object language in some suitable metalanguage.

It is proposed here to use the term *demonstrative* to mean *derivable, from the empty set of premises, in a sound deductive system* considering this to be a relation between sentences and semantics, a semantics being an account of the truth conditions for a language to which the sentence in question belongs.

It is normal practice among those who devise formal deductive systems to validate those systems by proving them sound. The effect of this (subject to some caveats which we will address later) is to ensure a connection between the things which are provable in these systems and the concept of analyticity.

To make this connection conspicuous I will give definitions of relevant concepts here from which the alleged elementary connection is readily seen.

I am concerned here exclusively with languages which have a well defined syntax and semantics. In order to make the desired connection it is convenient to think of the semantics as some kind of abstract entity, in fact, a function.

I will use the word *statement* here to mean an ordered pair of which the first element is a sentence and the second the semantics for a language in which that sentence is well-formed.

A statement will be called *demonstrative* if the sentence is derivable in a deductive system which is sound with respect to the semantics.

A *deductive system* is a set of sentences (the well formed sentences of the language of the deductive system) and an immediate-derivability relation, which is a relation between sets of sentences (the premises of an inference) and single sentences (the conclusion of an inference), all well formed sentences of the language. The *derivability* relation of a deductive system is the transitive closure of its immediate derivability relation.

A *semantics* is an assignment of meaning to the sentences of some language which is of interest here only insofar as it yields information about truth conditions for sentences in the language. We will model this as a function which assigns to each well formed sentence a meaning, the meaning being a set of *circumstances* under which that statement is true (these circumstances may be said to *satisfy* the sentence). The nature of a circumstance will vary from one language to the next, but might typically be a possible world and an assignment to free variables of entities in that possible world.

A deductive system is *sound* with respect to a semantics if it preserves satisfaction under the semantics, i.e. if all circumstances which

satisfy all the premises of a derivation under the semantics also satisfy the conclusion.

2.2. ANALYTIC TRUTH

2.2.1. *Prior Use*

2.2.2. *Definition*

The notion of analyticity likewise will be considered a relationship between sentences and semantics, a sentence being analytic (or an analytic truth) if its truth can be established from the semantics of the language alone, i.e. if the truth conditions show the sentence to be invariably true.

The term *analytic* will be used throughout as *analytic truth*.

A sentence in some well defined object language is *analytic* if it is satisfied under all circumstances.

2.3. DEMONSTRATIVE AND ANALYTIC ARE COEXTENSIVE

By an elementary induction on the length of proofs it follows that all the theorems of sound deductive systems are analytic. Every analytic sentence is demonstrative since it is provable in the sound deductive system which has just one inference rule whose conclusion is that sentence.

Hence:

Proposition 1. The concepts *analytic* and *demonstrative* (as defined) are coextensive.

We note that in particular under these definitions, the theorems of set theory (say ZFC) are demonstrative and analytic, and hence the theorems of mathematics in general.

3. Analyticity and Set Theory

Our next observation is that analyticity is reducible to, i.e. definable in terms of, set theoretic truth.

This is not an easily demonstrable claim. One reason for difficulty is that the universality of set theoretic truth makes that notion itself difficult to define. This matter will shortly be addressed, but in this preliminary attempt to justify interest in the concept via its connection with demonstrative truth no definition is available.

The justification of the claim is factored into two part. First it is alleged that for the purposes of determining the extension of analytic truth, abstract semantics suffices. Then the universality of set theory for abstract semantics is argued.

3.1. ABSTRACT SEMANTICS SUFFICES FOR DETERMINATION OF ANALYTIC TRUTH

3.2. THE UNIVERSALITY OF SET THEORY FOR ABSTRACT SEMANTICS

4. The Semantics of Set Theory

For the time being I propose to use this section for sundry discussions of set theory.

As well as there being various kinds of set theory (e.g. first order, second order, well-founded, non-well-founded, with or without a universal set), there are various different kinds of thing which might be offered as a semantics.

I am here concerned with set theory as a foundation for abstract semantics. Furthermore, my concern is with a foundation whose role is to provide a good response to the problem of semantic regress, rather than a foundation which is intended to provide a pragmatically convenient general context in which to undertake a formal development of some substantial body of demonstrative knowledge.

In meeting the latter need, which I hope to consider more fully in due course, non-well-founded set theories might possibly have a contribution to make. But for the former, well-founded set theory suffices.

When considering the semantics of well-founded set theory, we must first consider what set theoretic syntax is to be given meaning by the exercise.

4.1. THE ITERATIVE CONCEPTION OF SET

“The iterative conception of set” is the name given to a particular *explication* of the concept of a well-founded set (though not often presented as an account of a particular kind of set). It is generally held to have been articulated in the first instance in a paper by Zermelo dated about 1930

4.2. DEFINING TRUTH PREDICATES

Appendix

A. Extracts from FOM discussions

A.1. V DOES NOT EXIST

A.1.0.1. *The Proof* Fri Oct 7 03:52:27 EDT 2005

Both A.P.Hazen and Aatu Koskensisilta have responded to an argument on my part (though not mine) to the effect that the standard interpretation of V in NBG is incoherent.

Though I argued that calling V a class rather than a set would not escape the argument, Hazen felt that if V really were a different kind of thing:

"they are the (extensionalizations of) meanings of predicates of our set-theoretic language, and they exist only by being definable."

then my argument would fail.

Koskensisilta's response I didn't entirely understand, but seemed to be directed toward justifying quantification over classes, whereas my objection was not to quantification over classes. It was to the possibility of one particular class, V , being what it is supposed to be.

I provide below a new presentation of the argument which I think makes the argument more general and precise, and clarifies the character of the result.

The argument, as now presented is an argument about the concept "pure well-founded set" (which is what I take the iterative conception of set to be describing). It is to the effect that this concept does not have a "definite" extension.

The meaning of "definite" here is not crucial to the argument. In classical set theory as described in the iterative conception of set "definite" means something very weak (much weaker than the notion of "definite property" used in defining separation). It just means something like that the predicate or membership relation is boolean.

I offer the following definition:

Defn: A "pure well-founded set" is any definite collection of pure well-founded sets.

>From which I allege follows:

Lemma: Pure well-founded sets are pure and well-founded (in the usual sense of these terms).

My thesis is:

Theorem:

The extension of the concept "pure well-founded set" is not definite.

Proof: By reductio. Assume that it is definite and conclude that it both is and is not heteronymous.

Since the argument is about the concept of set itself, any object which purports to have a definite extension which coincides with that concept, however different that object may be from a set, must be tainted with the incoherence of supposing that the concept set has a definite extension.

For anyone who finds this argument too tenuously connected to the iterative conception of set, it can be reduced to something closer to that account via a similar argument to the effect that the extension of the concept ordinal (which corresponds of course to the stages in the iterative conception) cannot have a definite extension, and hence that the conception cannot describe a definite collection of sets.

A.1.0.2. *The Elaboration* Wed Oct 12 04:29:05 EDT 2005

On Saturday 08 October 2005 6:33 pm, Richard Heck wrote:

> >Both A.P.Hazen and Aatu Koskensilta have responded to an
> > argument on my part (though not mine) to the effect that the
> > standard interpretation of V in NBG is incoherent.
> >
> >Though I argued that calling V a class rather than a set

> > would not escape the argument, Hazen felt that if V really
 > > were a different kind of thing:
 > >
 > > "they are the (extensionalizations of) meanings
 > > of predicates of our set-theoretic language, and they
 > > exist only by being definable."
 > >
 > >then my argument would fail.
 >
 > Allen's language here is somewhat colorful, but I took his
 > point to rest upon the observation that quantification over
 > classes NBG can be understood as substitutional. Perhaps there
 > is a problem here I'm not remembering, one that is connected
 > with the presence of parameters in the comprehension axioms,
 > but I don't think so. In any event, much the same point could
 > be made in a different way: NBG can be interpreted in ZF(C)
 > plus a weak truth-theory, one in which the truth-predicate is
 > not allowed to figure in instances of schemata. If you think
 > of classes that way, then I think it's clear enough what
 > Allen's flourishes mean,

However, as I pointed out in my message, my argument is independent of the nature of V, speaking only to its intended extension, and has nothing to say about quantification over classes (though I could easily offer relevant corollaries).

I did not argue that NBG cannot be interpreted.

> and
 >
 > there is no conflict between NBG and the definition:
 > >Defn: A "pure well-founded set" is any definite collection of
 > > pure well-founded sets.
 >
 > which I take to be equivalent to Boolos's insistence that
 > set-theory is supposed to be about /all/ collections.

Well it certainly is not intended to be equivalent to it.

First of all, I don't see how a definition can be equivalent to an "insistence"!

If I were to take this alleged insistence as a definition

then I guess it would read "a set is any collection".

The difference between this and my own definition, which I will paraphrase for comparison as "a set is any definite collection of sets", seems to me very considerable.

My definition contains so much information that it runs very close to inconsistency (its a reductio absurdum on the possibility that the iterative conception could be completed).

The one you attribute to Boolos contains so little information that it runs close to vacuity.

My definition is a well-founded recursive definition. It is a definition by transfinite induction, and should be understood as involving the tacit codicil: nothing is a set unless its sethood is entailed by the definition.

>From the definition we are can derive a principle of transfinite induction asserting that sets have every "hereditary" property, where, in this context, a property is hereditary iff it is possessed by set whenever it is possessed by all its members.

Using this induction principle we can then prove that:

1. All sets are pure.
 2. All sets are well-founded.
- and hence
3. All sets are "heteronymous"
(i.e. do not contain themselves)

None of these conclusions flows from the insistance which you attribute to Boolos.

More controversially perhaps, it is plain from my definition that:

4. All definite collections of sets are sets.

and hence that there are no proper classes, unless something containing things other than sets or lacking a definite extension might be said to be a class.

Boolos's alleged insistence, would have the additional disadvantage that, taken out of context but with some knowledge of Boolos's metaphysics, we might reasonably interpret it as referring to all "actual" collections, where the meaning of "actual" if any, can only be discovered by probing Boolos's metaphysical intuitions.

By contrast, my definition may be understood as a definition, not of all the sets which "really exist" but as a definition of all the sets which might possibly exist, of which the sets intuited by Boolos are an infinitesimally small part.

A final but important difference between the "definitions" is the occurrence in mine of the concept "definite".

Without this the argument yields a contradiction without consideration of classes, suggesting that the concept of "set" is incoherent.

With it, it appears to demonstrate that there must be some characteristic of the extensions which yield sets which is not shared by the extensions which yield classes.

In my view it is best to read "definite" as a feature implicit in the first order formalisation of set theory, viz: that for any set s and any putative member x either x is in s or x is not in s .

This is of course, an instance of excluded middle.

For a theory to encompass collections which are not definite in this sense one would have to represent membership by something more complicated than a relation.

Possibly this motivates attempts to interpret classes as rules or formulae. However this won't help if the rule or formula or whatever, is still supposed to have a definite extension.

Since NBG is a first order language of set theory which includes "classes" such as V , this kind of "definiteness" of extension is possessed both by the sets and the classes, and the argument shows that the supposition that the extension of V is all the sets encompassed by the iterative conception (rather than all the sets in some other interpretation of NBG) is incoherent.

I guess that, even with this additional explanation you will not be convinced by this argument, and in that case I would be interested to know where you find the argument to be faulty.

For my part, coming across this particular definition of "set", (even though its an obvious definition of pure well-founded set and seems, obviously, to say the same thing as the iterative conception), has made a significant change to my beliefs about classes and about what kinds of accounts of the semantics for set theory are coherent.

I used to be suspicious about V, doubting whether the iterative conception of set could coherently be considered completeable.

But I knew of no argument for or against which I considered wholly convincing.

I now believe not only that the intended interpretation of NBG is incoherent, but also that formal set theories which do not mention classes cannot coherently be considered to be interpreted in the complete domain described by the iterative conception of set.

Of course, these are philosophical matters, so I don't imagine that these arguments are conclusive.

