Abstract Ontology

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Chapter 1

Preface

I have in mind here primarily an essay on those aspects of abstract ontology which have been of particular interest to me. I believe that might be more intelligible in the context of some history, and so I am including some historical notes.

Chapter 2

Substance

2.1 Introduction

Abstract entities are fundamental to deductive reasoning, to the foundations of mathematics and hence to large parts of science and engineering. More generally even than that (for all but radical scientists) they are fundamental to some conceptions of analytic method of the broadest scope.

Our starting point for this discussion is that abstract entities are in many respect like fictions, they are things which we assume to exist for various purposes, rather than things which we objectively discover. They are not *entirely* like fictions, for fictions are usually things which definitely don't exist (except possibly *as fictions*). Sherlock Homes is a detective featured in certain books, and I think it is true to say that there never was such a man, the things we say of him are fictitious and hence false, unless we speak *of the fiction* (as I do here) rather than *of the man.* By contrast, if in set theory we affirm the existence of the empty set, its not so obvious that we are asserting a falsehood.

The question whether abstract entities such as the empty set *really* exist is the kind of question which some philosophers have called *metaphysical.* It is a matter of controversy whether it has a definite meaning, but it is not my present concern.

For present purposes I put it to you that it is convenient in various ways (e.g. for the conduct of mathematics and hence for all those things which mathematics facilitates) to assume the existence of certain abstract objects, and I intend to discuss which such assumptions it is most advantageous to adopt. Or at least, I intend to describe to you the various deliberations on this topic in which I have engaged over the past quarter century, and some of the conclusions I have drawn from these deliberations.

2.2 The Role of Abstract Objects

The discussion of abstract ontology will be undertaken in the context of the following ideas about the role of abstract entities.

The broadest conception of that role can be seen in two distinct forms according to whether one takes a universalist or a pluralistic attitude towards language.

1A For modelling concrete (or other) systems.

1B For giving to languages an abstract semantics.

We think of *1A* as a description of the application of a single general purpose logical system to the great diversity of problems which can be subject to analysis by "nomologico-deductive" methods, i.e. by constructing an abstract model of some subject matter (which conforms to certain 'laws' observed in the target of the model).

We think of *1B* in the case that problem or domain oriented languages are adopted for modelling, or (in the manner espoused by Rudolf Carnap) when some special language has been devised for the purpose of formalising perhaps an entire scientific discipline. Deduction in these less universal languages may be underpinned by using abstract objects (possibly in some universalist context) to define an abstract semantics for the languages and thereby make definite the property of soundness essential to a deductive system for the language. Giving an abstract semantics to a language which is intended to talk about the concrete world need not, and normally would not be intended merely to provide a technical means for establishing consistency of a deductive system for that language. It would be normal for the chosen abstract ontology to mirror the relevant structure of the intended concrete subject matter in such a way as to enable a soundness proof relative to the abstract semantics to provide confidence in the soundness of the deductive system relative to the intended concrete subject matter.

The importance of abstract semantic is not confined to its role in supporting the credentials of formal languages with concrete interpretations. The realm of abstract entities is itself a subject matter in its own right, and may be said to be the subject matter of mathematics as a whole (though not all philosophers of mathematics would concur with this). Certain fundamental topics in mathematics can only be fully addressed if semantics is taken further in precision than is likely to be necessary for any concrete application. An important topic of this kind is cardinal arithmetic, in which it is now common to consider meaningless question which are not decided by the first order axioms known as ZFC. However, if it is accepted that certain informal semantic notions are sufficiently definite, in particular the notion of "all subsets", then cardinal arithmetic becomes categorical and many important questions (such as the Continuum Hypothesis and the Generalised Continuum Hypothesis) are thereby rendered meaningful. This procedure is an extension of the idea that "true arithmetic" (the sentences of the first order theory of arithmetic which are true in relation to the natural numbers) is well defined even though it can have no complete deductive system.

These two perspectives are by no means distinct, the suggested pluralism may be universalistic in relation to semantic foundations. The two are complementary, and there are pragmatic approaches which fall at various points between the two, among which important examples are methods for supporting pluralism by semantic embedding in universalistic frameworks.

These general perspectives on the role of abstract entities can be supplemented by identifying a number of slightly more specific roles.

They are:

1. For establishing the consistency of definitions.
2. For comparing structures.
3. For re-use of abstract ideas and the large scale structuring of specifications or theories.

modelling Certain important kinds of knowledge can reasonably be presented as consisting in our having abstract models of various aspects of the concrete world. The value in such abstract models is in permitting us to anticipate the consequences of future actions and with this information to chose those courses of action whose outcomes we prefer.

The advantage of abstract models in this context is in their precision. This enables us to reason reliably about their characteristics and about the behaviour of any real phenomena of which they provide a good model.

In constructing such a model abstract objects serve as surrogates for the concrete entities in the real world (or of whatever kind of entities we might want to reason about).

consistency This kind of modelling is often clearly *mathematical* (sometimes not) and there is a second useful perspective on the role of abstract objects in this context. In the logical development of mathematics, various mathematical concepts are defined and the consequences of the definitions are then explored by deduction. It is important that what we say in a definition is *consistent*, for otherwise any conclusion may flow from the definitions. The usual method used to establish consistency of such definitions is to exhibit something which satisfies the definition, and a substantial abstract ontology is convenient for this purpose.

comparisons In order to build and reason about abstract models of the concrete world it is helpful to develop abstract theories of the various kinds of abstract entities which appear in such models. In the making of such comparisons abstract objects have a special role. The correspondences between abstract entities which feature in such comparisons are themselves abstract objects, and an insufficient ontology may cause a comparison to fail through no pertinent dissimilarity in the objects compared.

re-use The problem of *re-use* arises from constraints which have been imposed in logical system to ensure the consistency of the logic. Typically these are constraints on abstraction, and hence on the range of abstract objects which can be shown to exist, which are intended to avoid incoherent ontology but which also eliminate some consistent and convenient objects and impose difficulties in abstract mathematics. Perhaps the best publicised of these is now the difficulties arising in category in relation to categories embracing all algebras with a certain signature, or even the category of categories.

2.3 Well-Founded Sets

2.4 Sets

An ontology of well-founded sets probably does provide the best context to address the most critical foundational issue, that of consistency.

However, in the conduct of mathematics we need in the course of investigating the various kinds of abstract entity.we need certain other abstract entities to be fulfil auxiliary roles in our investigation.

2.4.1 Pragmatic Considerations

2.4.2 The Problem of Consistency

2.4.3 The Construction of Models

Chapter 3

History

In the first instance this part will consist of no more than a sketch of the modern history as I understand it.

This will cover relevant aspects of the work of:

- Cantor
- Frege
- Russell
- Hilbert
- Carnap
- Schonfinkel
- Quine
- Church
- Curry
- Woodin

The historical sketch will evolve and be filled out as necessary to cover those aspects of the history which are relevant background to my principle narrative on abstract ontology.

For two millennia the mathematics of magnitudes developed with a substantially incomplete understanding of the nature of these magnitudes. The invention of the infinitesimal calculus inaugurated a new era involving new conceptual difficulties in form of infinitesimal magnitudes alongside a considerable expansion in the development of analysis for application in science. Despite foundational weakness these developments were highly successful, and it was not until the nineteenth century that mathematicians addressed the foundational issues and put analysis on a sound basis. This was done first by the elimination of infinitesimals (later shown not to be strictly necessary, but nevertheless a valuable simplification), and then by a precise definition of a number system adequate for the mathematics of magnitude, by construction in stages from the natural numbers.

3.1 Cantor

Chapter 4

Further Notes

4.1 Hallett on Cantor

This section concerns issues arising for me from Hallett's book *Cantorian Set Theory and Limitation of Size*[**?**].

Why I am Looking at This Book There is a very great difference in perspective between myself of Hallett on set theory, which it might be illuminating to explore. To draw this out I speak here of what I am hoping to find in his book, and how this differs from what he tried to deliver.

My dominant interest is in understanding abstract ontology, because of its foundational significance in the application of abstract modelling in nomologico-deductive methods generally.

Drawing Out the Differences I'm going to scrutinise some minutiae from very early in the book

4.1.1 Preface

Hallett begins with the question whether set theory is simple or not. It is his point that it is not so simple.

I think he makes it seem less simple than it is. In doing this he is at one with most philosophers or philosophically minded logicians and set theorists.

I think it is important to discriminate between various different aspects of set theory, some of which are very difficult and complex, and others of which really are very simple.

In examining these distinctions I begin by saying that the notion of set which I intend to discuss here is the notion of a pure well-founded set. In this it is useful to distinguish two questions:

1. What is a set?

2. What sets are there?

The first question does have a simple answer. A set is something whose essence consists in the having of members, and in the members it has. This is captured by two features its first order formalisations. The first is that membership is a binary relation between sets, so that of any two sets it is either true or it is false that the first is a member of the second. The second is the axiom of extensionality, which tells us that no two distinct sets have exactly the same members. From this we discover that there is in the essence of a set nothing beyond its members, no supplementary objects or information, no particulars about how the members are arranged in the set of which they are members.

Sets, by themselves are not very useful. It is only when we have quite a lot of them that we find sets to be useful. A "set theory" is a theory in which the objects in the domain of discourse are sets.

4.2 Dehornoy on Woodin on CH

The following are my notes on reading the Dehornoy's paper Recent Progress on the Continuum Hypothesis (after Woodin). The points I raise may not of course be relevant to Woodin, they points may only connect with Dehornoy's exposition.

4.2.1 Conjecture 1

First note the way in which Dehornoy introduces conjecture 1:

> "For the first time, there appear a global explanation for the hierarchy of large cardinals, and, chiefly, a realistic perspective to decide the Continuum Hypothesis namely in the negative."

and this is the conjecture:

> "Every set theory that is compatible with the existence of large cardinals and makes the properties of sets with hereditary cardinality at most $\aleph_1$ invariant under forcing implies that the Continuum Hypothesis be false."

I don't actually understand this conjecture. Perhaps by the end of the paper I will, but here are some indications of what is unclear to me about it at this stage.

The existence of large cardinals

Does "compatible with the existence of large cardinals" in relation to a set theory mean that the theory has models in which large cardinals exist (for every possible cardinality?), or does it mean that it does not contradict any large cardinal axiom?

If the first, I will assume that the intention is that for every large cardinal there exist a model in which that large cardinal appears. There remains a question about what it means to say that a certain large cardinal exists in some interpretation of set theory.

It is probably reasonable to assume that the notion of cardinality involved here is internal, i.e. cardinality in an interpretation of set theory is determined by reference to the available bijections in that interpretation rather than in some other (e.g. a standard interpretation, to give an absolute notion of cardinality).

At the end of this section Dehornoy explains the idea of axioms, A, being compatible with the existence of large cardinals as "in the sense that no large cardinal axiom contradicts A". This leaves open the question, "what is a large cardinal axiom"? This point arises in connection with Dehornoy's view that it is reasonable to consider large cardinal axioms true, or at least to doubt any axiom which is contradicted by a large cardinal axiom.

For my part, I consider the explication of the notion of large cardinal axiom crucial to the case for considering them to be justified by the iterative conception of set. It is essential that a large cardinal axiom be in some sense “pure”, in doing nothing but placing a lower bound on the size of the largest cardinal. An obvious way to get an “impure” large cardinal axiom would be to conjoin a large cardinal with CH or its negation. One needs in some way to be reassured of any putative large cardinal principle that it does not covertly impose other conditions, before accepting its truth in some initial segment of the cumulative hierarchy. Furthermore, to prove results in which the phrase “all large cardinal axioms” appear, it would seem to be essential to settle the meaning of that phrase.

properties invariant under forcing

Implies that the Continuum Hypothesis be false

Presumably this means “includes the denial of the continuum hypothesis” (in the theory).

Further remarks

While talking about the realistic perspective, Dehornoy nevertheless conceives of his problem as finding appropriate axiomatisations of set theory, rather than simply establishing the truth of CH (though that may be the proximate aim of the axiomatisation). Feodora, oddly, he talks about axioms possibly “completing” ZFC, which of course we know to be impossible, at least in the usual sense, or even in the limited sense of making ZFC arithmetically complete. At this point I can only suppose, not that he is unaware of this point but rather that he is using the term to mean some further extension of ZFC which might be considered *sufficiently* complete for some purposes.

4.3 Arithmetic, Incompleteness and Forcing

Here Dehornoy begins “let V be the collection of all sets”! Of course, there is no such collection.

In a footnote he qualifies this as just meaning the “pure” sets. He doesn’t add the constraint “well founded”, but he gives a definition which does seem to entail well-foundedness. However, there can be no such collection, for that collection would be a set according to the definition, and would be well-founded, and hence could not be the collection of all pure well-founded sets.

Later, even more oddly after this definition, we find talk about generic extensions of V.

Dehornoy now talks about the incompleteness of ZFC and makes a distinction between the kinds of incompleteness which correspond to and are demonstrated by Gödel’s first incompleteness result, and other “higher level” undecided statements such as the continuum hypothesis. The distinction is found in forcing, and in the explanation we begin with the definition of “invariant under forcing”.

> Let H be a definable set. We say that the properties of the structure (H, $\in$) are invariant under forcing if, for every sentence ψ, every model M, and every generic extension M[G] of M, the sentence “(H, $\in$) satisfies ψ” is satisfied in M if and only if it is satisfied in M[G].

A footnote defines H as “definable” if it is the set of values x which satisfy some formula $\psi(x)$. Let us assume that this is to be understood as *in V*.

This is not what one might naturally expect, and I would have been inclined to use more specific terminology to describe this condition. Where the phrase “makes the properties of certain sets invariant under forcing” is used in conjecture 1, I would be inclined to say “makes the formulae true in certain membership structures invariant under forcing”. [this isn’t right]

4.4 Peter Koellner

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